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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

CRITERIA FOR THE IRREDUCIBILITY OF FUNCTIONS IN A FINITE FIELD.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 3, 1906.)

1. THEOREM. *A necessary condition that*

$$f(x) \equiv x^m + c_1 x^{m-1} + \cdots + c_m \tag{1}$$

shall be irreducible in the GF $[p^n]$, $p > 2$, *is that its discriminant* * *be a square or a not-square according as* m *is odd or even.*

If $f(x) = 0$ is irreducible its roots are $\lambda^{p^i}(i = 0, 1, \cdots, m-1)$. Its discriminant is therefore the square of P, where

$$P = \prod_{i<j}^{i,j=0,1,\ldots,m-1} (\lambda^{p^i} - \lambda^{p^j}) \equiv \Pi f_{i,j}.$$

For $j < m-1$, we have $f_{i,j}^{p^n} = f_{i+1,j+1}$. But $f_{i,m-1}^{p^n} = -f_{0,i+1}$. Hence

$$P^{p^n} = (-1)^{m-1} P,$$

so that P equals a mark of the $GF[p^n]$, $p > 2$, if and only if m is odd.

Remark. The condition is also sufficient if $m = 2$.

2. LEMMA. *The necessary and sufficient condition that a cubic shall have one and but one root in the* $GF[p^n]$, $p > 2$, *is that its discriminant be a not-square.*

---

* As in the theory of algebraic equations, it is here convenient to designate as the discriminant the product of the squares of the differences of the roots. Most writers on cognate subjects insert the factor $(-1)^{\frac{1}{2}m(m-1)}$, and some insert also the factor $1/m^m$.

The condition is necessary. For, if the cubic has a linear and an irreducible quadratic factor, its roots are

$$y_1 \equiv y_1^{p^n}, \quad y_2, \quad y_3 \equiv y_2^{p^n} \quad (y_2^{p^{2n}} = y_2).$$

Then its discriminant Δ equals π^2, where

$$\pi = (y_1 - y_2)(y_1 - y_2^{p^n})(y_2 - y_2^{p^n}), \quad \pi^{p^n} = -\pi.$$

Since π is not in the $GF[p^n]$, $p > 2$, Δ is a not-square.

The condition is sufficient. For, if Δ is a not-square, the cubic is reducible (§ 1). But not all three roots lie in the $GF[p^n]$, since Δ would then be a square in the field.

3. THEOREM.* *The necessary and sufficient conditions that*

$$x^3 + \beta x + b = 0 \tag{2}$$

be irreducible in the $GF[p^n]$, $p > 3$, *are the following two:*

(3) $R \equiv -4\beta^3 - 27b^2 = a$ *square* $\neq 0$ *in* $GF[p^n]$, *say* $R = 81\mu^2$;

(4) $\frac{1}{2}(-b + \mu\sqrt{-3}) = a$ *not-cube in the field* $(GF[p^n], \sqrt{-3})$.

We suppose that the necessary condition (3) is satisfied. In (2) set $x = y - \beta/3y$. Then

$$(2y^3 + b)^2 = -R/27 = -3\mu^2. \tag{5}$$

Now -3 is a square or a not-square in the $GF[p^n]$, $p > 3$, according as $(x^3 - 1)/(x - 1) = 0$ is or is not solvable, *i. e.*, according as $p^n - 1$ is or is not divisible by 3.

First, let p^n be of the form $3l + 1$, so that -3 is the square of a mark τ in the $GF[p^n]$. Then (5) gives $y^3 = \frac{1}{2}(-b + \mu\tau)$. If the second member were a cube in the field, (2) would have a root in the field. Hence (4) is a necessary condition for irreducibility. Further, it is a sufficient condition. For, if $\frac{1}{2}(-b + \mu\tau)$ is a not-cube ν, $y^3 = \nu$ is irreducible in the $GF[p^n]$. The same is true of (2). Indeed, if it had a root x in the field, then y would satisfy the quadratic $x = y - \beta y^2/3\nu$, in contradiction to the irreducibility of $y^3 = \nu$.

* From §§ 2, 3, we deduce complete criteria for the nature of the roots of a cubic.

Next, let p^n be of the form $3l+2$, so that $\tau^2=-3$ defines the $GF[p^{2n}]$. If

$$\tfrac{1}{2}(-b+\mu\tau)=(r+s\tau)^3, \quad r \text{ and } s \text{ in the } GF[p^n],$$

then

$$\tfrac{1}{2}(-b-\mu\tau)=(r-s\tau)^3.$$

The product of the left members equals $-\beta^3/27$. Hence

$$y=r+s\tau, \quad -\beta/3y=r-s\tau, \quad x=2r,$$

so that (2) would be reducible. To show that (4) is also a sufficient condition, let $\frac{1}{2}(-b+\mu\tau)$ be a not-cube ν in the $GF[p^{2n}]$. Then if (2) had a root x in the $GF[p^{2n}]$, y would satisfy two equations with coefficients in this field, $x=y-\beta y^2/3\nu$, $y^3=\nu$, whereas the latter is irreducible.

4. When $p^n=3l+2$, there are $\frac{2}{3}(p^{2n}-1)$ not-cubes * ν in the $GF[p^{2n}]$. For a given ν, marks b and β of the $GF[p^n]$ are uniquely determined by the condition $\frac{1}{2}(-b+\mu\sqrt{-3})=\nu$, viz.,

$$-b=\nu+\nu^{p^n}, \quad -\beta^3/27=\nu^{p^n+1}.$$

By the latter, $\beta=-3f\nu^{l+1}$, where $f^3=1$. Then $\beta^{p^n}=\beta$ if and only if

$$f\nu^{\frac{1}{2}(p^{2n}-1)}=1, \quad \beta=-3\nu^{-\frac{1}{2}(p^n-2)(p^n+1)}.$$

The values of b and β are unaltered by the replacement of ν by either ν or ν^{p^n}, but at least one is altered by any new replacement. *When $p^n=3l+2$, there exist exactly $\frac{1}{3}(p^{2n}-1)$ irreducible cubics (2) in the $GF[p^n]$; these are given by*

$$(6)\quad x^3-3x\nu^{-\frac{1}{2}(p^n-2)(p^n+1)}-\nu-\nu^{p^n}=0 \quad (\nu \text{ a not-cube in } GF[p^{2n}]).$$

5. When $p^n=3l+1$, (4) requires that

$$\tfrac{1}{2}(-b+\mu\sqrt{-3})=\nu, \quad \tfrac{1}{2}(b+\mu\sqrt{-3})=\beta^3/27\nu \quad (\nu \text{ a not-cube}),$$

since the product of the first members is $\beta^3/27$. Hence

$$b=\beta^3/27\nu-\nu, \quad \mu\sqrt{-3}=\beta^3/27\nu+\nu.$$

* The powers of a primitive root with exponents prime to 3.

For these values, (3) is satisfied. Now b is unaltered by the replacement of ν by either ν or $-\beta^3/27\nu$, but is altered by any new replacement. Separating the cases $\beta \neq 0$, $\beta = 0$, we obtain

$$\tfrac{1}{2}\cdot\tfrac{2}{3}(p^n-1)(p^n-1)+\tfrac{2}{3}(p^n-1)\equiv\tfrac{1}{3}(p^{2n}-1)$$

sets ν, β giving distinct sets b, β. *When* $p^n = 3l+1$, *there exist exactly* $\frac{1}{3}(p^{2n}-1)$ *irreducible cubics* (2) *in the* $GF[p^n]$; *these are given by*

$$(7)\qquad x^3+\beta x+\beta^3/27\nu-\nu=0 \qquad (\beta \textit{ arbitrary, } \nu \textit{ not-cube}).$$

6. We may otherwise determine the number of irreducible cubics in the $GF[p^n]$. We enumerate the reducible cubics (2).

First, let there be three linear factors $x-f_j$, where $f_1+f_2+f_3=0$. Consider in turn the cases: (*i*) every $f_j=0$; (*ii*) a single $f_j=0$; (*iii*) each $f_j\neq 0$, two equal; (*iv*) each $f_j\neq 0$, all distinct. For the first three cases, the number of cubics is

$$1+\tfrac{1}{2}(p^n-1)+(p^n-1) \text{ if } p>2;\ 1+(p^n-1)+0 \text{ if } p=2.$$

For case (*iv*) it suffices to take as f_1 any mark $\neq 0$, for f_2 any mark distinct from $0, f_1, -f_1, -2f_1, -\frac{1}{2}f_1$, the last three not occurring if $p=2$, and the last two being superfluous if $p=3$. The number of cubics for case (*iv*) is thus

$$\tfrac{1}{6}(p^n-1)(p^n-5) \text{ if } p>3,\quad \tfrac{1}{6}(p^n-1)(p^n-3) \text{ if } p=3,$$
$$\tfrac{1}{6}(p^n-1)(p^n-2) \text{ if } p=2.$$

The number of cubics for cases (*i*)–(*iv*) is therefore

$$1+\tfrac{1}{6}(p^n-1)(p^n+4) \text{ if } p\neq 3,\quad 1+\tfrac{1}{6}(p^n-1)(p^n+6) \text{ if } p=3.$$

Finally, let the function be $(x-f)(x^2+fx+g)$, where the last factor is irreducible. The number of these functions is $\frac{1}{2}p^n(p^n-1)$, since, of the reducible quadratics, p^n have equal roots and $\frac{1}{2}p^n(p^n-1)$ have distinct roots.

Hence the total number of reducible cubics (2) is

$$\tfrac{1}{3}(2p^{2n}+1) \text{ if } p\neq 3;\quad \tfrac{1}{3}(2p^{2n}+p^n) \text{ if } p=3.$$

But the total number of cubics (2) is p^{2n}. *Hence the number of irreducible functions* $x^3+\beta x+b$ *in the* $GF[p^n]$ *is* $\frac{1}{3}(p^{2n}-1)$ *if* $p\neq 3$, *and* $\frac{1}{3}(3^{2n}-3^n)$ *if* $p=3$.

7. It remains to exhibit the $\frac{1}{3}(3^{2n} - 3^n)$ irreducible cubics (2) in the $GF[3^n]$. It is known * that

$$x^p - x/\lambda^{p-1} - \beta/\lambda^p \tag{8}$$

is irreducible in the $GF[p^n]$ if, and only if,

$$\beta^{p^{n-1}} + \beta^{p^{n-2}} + \cdots + \beta^p + \beta \neq 0. \tag{9}$$

Now if $\lambda^{p-1} = \lambda_1^{p-1}$, $\beta/\lambda^p = \beta_1\lambda_1^p$, then $\lambda_1 = \epsilon\lambda$, $\beta_1 = \epsilon\beta$, where $\epsilon^{p-1} = 1$. But if β satisfies condition (9), then also $\epsilon\beta$ does. Hence the number of distinct irreducible functions (8) is

$$(p^n - 1)(p^n - p^{n-1}) \div (p - 1) \equiv p^{2n-1} - p^{n-1}.$$

Hence for $p = 3$ every irreducible cubic (2) is given by (8), (9).

8. Consider next the quartic

$$x^4 + ax^3 + bx^2 + cx + d = 0. \tag{10}$$

Its discriminant equals that of the resolvent cubic

$$y^3 - by^2 + (ac - 4d)y + 4bd - a^2d - c^2 = 0. \tag{11}$$

When (10) is irreducible in the $GF[p^n]$, its discriminant is a not-square (§1), so that (11) is reducible. To verify this result, let

$$\lambda, \quad \lambda^{p^n}, \quad \lambda^{p^{2n}}, \quad \lambda^{p^{3n}} \qquad (\lambda^{p^{4n}} = \lambda)$$

be the roots of (10). Then (11) has the roots

$$y_1 = \lambda^{p^{2n}+1} + \lambda^{p^{3n}+p^n}, \; y_2 = \lambda^{p^n+1} + \lambda^{p^{2n}+p^{3n}}, \; y_3 = \lambda^{p^{2n}+1} + \lambda^{p^{3n}+p^n}.$$

Hence $y_1^{p^n} = y_1$, $y_2^{p^n} = y_3$, $y_3^{p^n} = y_2$. But $y_2 \neq y_3$. *Hence when* (10) *is irreducible in the* $GF[p^n]$, *its resolvent cubic has a single root in the field.*

Now (11) may be written in the form

$$(\tfrac{1}{4}y^2 - d)(a^2 - 4b + 4y) = (\tfrac{1}{2}ay - c)^2. \tag{11'}$$

But $a^2 - 4b + 4y_1 = t^2$, where

$$t = \lambda + \lambda^{p^{2n}} - \lambda^{p^n} - \lambda^{p^{3n}}, \quad t^{p^n} = -t.$$

Hence t is not in the $GF[p^n]$, $p > 2$, unless $t = 0$.

* Dickson, Linear Groups, p. 29.

9. We first dispose of the special case $t = 0$. Then, by (11′),

$$a^2 - 4b + 4y_1 = 0, \quad \tfrac{1}{2}ay_1 = c.$$

Eliminating y_1, we get

$$c = \tfrac{1}{2}ab - \tfrac{1}{8}a^3. \tag{12}$$

Then the quartic (10) may be written in the form

$$(x^2 + \tfrac{1}{2}ax + \tfrac{1}{2}b - \tfrac{1}{8}a^2)^2 = k, \quad k \equiv (\tfrac{1}{2}b - \tfrac{1}{8}a^2)^2 - d. \tag{13}$$

Then k must be a not-square; otherwise, the quartic would have two quadratic factors with coefficients in the $GF[p^n]$.

Suppose that this condition on k is satisfied. In view of (13), the quartic has no root in the $GF[p^n]$. It remains to find the further conditions in order that (10) shall not be the product of two quadratic factors in the field. Such factors, when existing, must be of the form

$$x^2 + (\tfrac{1}{2}a + r)x \pm s, \quad x^2 + (\tfrac{1}{2}a - r)x + ds^{-1},$$

where

$$s + ds^{-1} + \tfrac{1}{4}a^2 - r^2 = b, \quad s(\tfrac{1}{2}a - r) + ds^{-1}(\tfrac{1}{2}a + r) = c. \tag{14}$$

Multiplying the first by $\frac{1}{2}a$ and subtracting the result from the second, we obtain, in view of (12),

$$r(ds^{-1} - s + \tfrac{1}{2}ar) = 0. \tag{15}$$

If $r = 0$, (14_1) becomes $(2s + \frac{1}{2}a^2 - b)^2 = 4k$. Hence $r \neq 0$. If $a = 0$, then $s^2 = d$. If d is a square, (14_1) is satisfied when r^2 equals $\pm 2\sqrt{d} - b$; but one of these values is a square, since their product equals the not-square $4k$. The result for this case is stated in the corollary below. Let next $a \neq 0$. If we eliminate r between (15) and (14_1), we obtain after simple modifications

$$\left\{\frac{2}{a}\left(s + \frac{d}{s}\right) - \frac{a}{4}\right\}^2 = w, \qquad w \equiv \frac{5}{16}a^2 - b + 16d/a^2.$$

There exist solutions s in the $GF[p^n]$ if and only if w is a square. In fact, when w is a square, s lies in the field if

$$(\tfrac{1}{8}a^2 \pm \tfrac{1}{2}aw^{\frac{1}{2}})^2 - 4d$$

is a square. But the product of these two expressions is seen to equal the not-square $\frac{1}{4}a^4k$, so that one is a square.

THEOREM. *If the coefficients of the quartic* (10) *satisfy* (12), *it is irreducible in the* $GF[p^n]$, $p>2$, *if and only if* $(\frac{1}{2}b-\frac{1}{8}a^2)^2 - d$ *and* $\frac{5}{16}a^4 - a^2b + 16d$ *are not-squares.*

Corollary. *In the* $GF[p^n]$, $p>2$, x^4+bx^2+d *is irreducible if and only if* b^2-4d *and* d *are not-squares.*

10. THEOREM. *The necessary and sufficient conditions that a quartic* (10), *not satisfying* (12), *be irreducible in the* $GF[p^n]$, $p>2$, *are that the resolvent cubic shall have* one and but one root* y_1 *in the* $GF[p^n]$ *and that* $a^2-4b+4y_1$ *shall be a not-square.*

These conditions are necessary by § 8 since now $t \neq 0$.

It remains to prove that these conditions are sufficient. Since there are three ways of separating the four roots into pairs, there are exactly three sets of two quadratic factors of (10), viz.,

$$x^2 + \tfrac{1}{2}(a \mp \tfrac{1}{2}t_i)x + \tfrac{1}{2}y_i \mp (\tfrac{1}{2}ay_i - c)/t_i = 0 \quad (t_i^2 = y_i).$$

Note that under the assumptions each $t_i \neq 0$. If a pair of factors have as coefficients marks of the $GF[p^n]$, then y_i and t_i belong to the field. But y_2, y_3 and t_1 are not in the $GF[p^n]$. Hence there is no decomposition into quadratic factors in the field. Finally, we show that there is no linear factor. The quartic may be written in the form

$$X^2 = (a^2-4b+4y_1)Z^2, \quad X \equiv x^2 + \tfrac{1}{2}ax + \tfrac{1}{2}y_1,$$

$$Y \equiv \tfrac{1}{2}x + \frac{\frac{1}{2}ay_1 - c}{a^2-4b+4y_1}.$$

If the quartic has a root x in the field, then $X=Z=0$. To prove that this is impossible it suffices to set $a=0$, as may be accomplished by a transformation of the quartic. Then

$$x^2 + \tfrac{1}{2}y_1 = 0, \quad \tfrac{1}{2}x - c/(4y_1 - 4b) = 0$$

require that

$$y_1^3 - 2by_1^2 + b^2y_1 + \tfrac{1}{2}c^2 = 0. \tag{16}$$

* We may replace this condition by the condition that Δ shall be a not-square (§ 2).

On eliminating first y_1^3 and then c^2 between (16) and (11), and in the latter case removing the factor $y_1 - b \neq 0$, we get

$$(17)\quad by_1^2 - b^2y_1 - 4dy_1 + 4bd - \tfrac{3}{2}c^2 = 0, \quad 3y_1^2 - 2by_1 - 4d = 0.$$

But the Sylvester eliminant of these is found to equal $-\frac{3}{4}\Delta$, where Δ is the discriminant of (11) for $a = 0$. Hence the case $p \neq 3$ is excluded. For $p = 3$, $by_1 = d \neq 0$; but for this value of y_1 (11) becomes $\Delta = 0$. Hence in every case the conditions are sufficient to make the quartic irreducible.

UNIVERSITY OF CHICAGO
June, 1906.

ON THE THEORY OF EQUATIONS IN A MODULAR FIELD.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 3, 1906.)

1. The object of this note is to point out that the Galois theory of algebraic equations may be extended to equations in a field F having a modulus p. For a finite field the theory is so obvious that this case furnishes a simple, but characteristic, example of the Galois theory.

2. Let the domain of rationality be the $GF[p^n]$. Consider an equation $f(x) = 0$ with coefficients in this field and having distinct roots $x_1, \cdots, x_m$. By a Galois resolvent of $f(x) = 0$ will be meant an equation $\phi(V) = 0$, irreducible in the $GF[p^n]$, with a root V which is a rational function of $x_1, \cdots, x_m$, and such that each x_i is a rational function of V with coefficients in the field. Let $m_1, m_2, \cdots$ be the distinct degrees of the irreducible factors of $f(x)$ in the $GF[p^n]$, and let l be the least common multiple of $m_1, m_2, \cdots$. Then the smallest field which contains all the roots x_i of $f(x) = 0$ is the $GF[p^{nl}]$. Any primitive root of the latter may be taken as the desired function V. In fact the power $(p^{nl} - 1)/(p^{nm_i} - 1)$ of V is a primitive root ρ_i of the $GF[p^{nm_i}]$, so that the x's are powers of $\rho_1, \rho_2, \cdots$ and hence of V. Further, V is a rational function of the ρ's and hence of the x's. Indeed, if m_1 and m_2 have the greatest common divisor g and the least common multiple λ, an

irreducible equation of degree m_1 decomposes in the $GF[p^{n m_2}]$ into g irreducible equations, any one of which defines the $GF[p^{n\lambda}]$, so that the marks of the latter are rational functions of ρ_1 and ρ_2.

The Galois resolvent $\phi(V) = 0$ has the m roots

$$V^{p^{ni}} \qquad (i = 0, 1, \cdots, m-1).$$

Hence the Galois group of $f(x) = 0$ in the $GF[p^n]$ is simply isomorphic with the cyclic group of order m generated by the substitution

$$V' = V^{p^n}.$$

3. One standard method (following Galois) of obtaining the Galois resolvent of an algebraic equation of degree m in a domain of rationality R depends upon the existence of an $m!$ valued rational function V of the roots x_i. When R is the $GF[p^n]$, this method cannot be applied unless p^n exceeds a certain limit. For example, if all the x's belong to the $GF[p^n]$, V evidently does not exist when $m! > p^n$. Again, if the irreducible factors of $f(x)$ are all of degree $\leqq 2$, V does not exist when $m! > p^{2n}$. Except for certain values of p^n, small in relation to m, there exists an $m!$ valued function V and the theory presents no difficulty.*

4. There is an interesting point in the determination of an $m!$ valued linear function of the distinct roots $x_1, \cdots, x_m$. It is illustrated in the case $m = 3$. If $t = \alpha x_1 + \beta x_2 + \gamma x_3$ is six valued, so is $t - \gamma(x_1 + x_2 + x_3)$ six valued. Hence we may restrict our attention to $V = qx_1 + x_2$. The necessary and sufficient condition that V be six valued is that q be distinct from

$$0,\ 1,\ (x_2 - x_3)/(x_1 - x_3),\ (x_1 - x_2)/(x_1 - x_3),\ (x_3 - x_2)/(x_1 - x_2),$$

and the reciprocals of the last three. Denoting the first of these fractions by λ, we find that the other two are $1 - \lambda$ and

* In a discussion of the Galois group of order N (*cf.* Dickson's Algebraic Equations, p. 53), there is a proof that if $\phi_1 = \phi_2 = \cdots = \phi_N$, then $\phi_1 = N^{-1}(\phi_1 + \cdots + \phi_N) = N^{-1}$ (a number of R). This step would now be invalid if N is a multiple of the modulus p. But we may proceed as follows: Let $N = p^s k$, k prime to p. Take the ${}_N C_p = p^{s-1} k_1$ products π_i of the ϕ's p at a time; take the $p^{s-2} k_2$ products π'_i of the π_i p at a time; etc. Finally, we reach $k_s \phi^{p^s}$ as a symmetric function of the ϕ_i. But k_s is prime to p, and the extraction of (p^s)th roots is possible in the field.

$\lambda/(\lambda-1)$. Hence

$$(1)\quad q \neq 0,\ 1,\ \lambda,\ 1-\lambda,\ \frac{\lambda}{\lambda-1},\ \frac{1}{\lambda},\ \frac{1}{1-\lambda},\ \frac{\lambda-1}{\lambda}\quad (\lambda \neq 0,\ 1).$$

Here the six functions of λ are the six elements of the cross-ratio group, and each differs from 0 and 1. Hence equalities arise only when $\lambda = -1,\ 2$, or $\frac{1}{2}$, or when $\lambda^2 - \lambda + 1 = 0$.

When x_1, x_2, x_3 are distinct, $qx_1 + x_2$ is six valued only in the following cases: (*i*) *one of the x's is an arithmetical mean between the other two, with $q \neq 0,\ 1,\ -1,\ 2,\ \frac{1}{2}$;* (*ii*) $\Sigma x_j^2 = \Sigma x_j x_k$, *with $q \neq 0,\ 1,\ \lambda,\ 1/\lambda$ (where $\lambda^2 - \lambda + 1 = 0$);* (*iii*) *neither of the relations on the x's holding, with q not equal to one of the eight distinct values* (1).

It may now be readily shown that there exist six valued linear functions of the roots x_i of a cubic in the $GF[p^n]$ when $p^n > 8$; when $p^n = 7$; and when $p^n = 5$ or 8, with the x_i not all in the $GF[p^n]$.

5. In conclusion it may be remarked that the Galois theory as presented in Weber's Algebra may readily be extended to apply to modular fields, provided his argument on page 500 (of volume 1 of the second edition) be replaced by that in § 2 above.

THE UNIVERSITY OF CHICAGO,
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NOTE ON THE VARIATION OF THE DEFINITE INTEGRAL.

BY MR. N. J. LENNES.

(Read before the Chicago Section of the American Mathematical Society. April 14, 1906.)

A function is said to be of limited variation on an interval ab if the set of sums

$$\left[\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|\right]$$

is bounded for the set of all partitions of ab. The points $a = x_0, x_1, x_2 \cdots, x_{n-1}, x_n = b$ of each partition are ordered on the interval according to the subscripts. The least upper

bound of this set of sums is denoted by $V_a^b f(x)$ and is called the variation of $f(x)$ on ab. We consider the variation of

$$\int_a^x f(x)dx$$

on an interval ab, where

$$\int_a^x f(x)dx$$

is a function of the upper limit of integration. Denote this variation by

$$V_a^b \int_a^x f(x)dx.$$

THEOREM I. *If $f(x)$ does not change sign on ab then*

$$V_a^b \int_a^x f(x)dx$$

exists on ab and

$$V_a^b \int_a^x f(x)dx = \int_a^b |f(x)|\,dx.$$

Proof: The theorem is obvious since for every partition of ab consisting of $n+1$ points

$$\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx \right| = \int_a^b |f(x)|\,dx.$$

For convenience of reference a definition of the definite integral is inserted. Let ab be an interval upon which a function $f(x)$ is defined, single valued and bounded. Let π_δ stand for any partition of ab consisting of the points $x_0 = a, x_1, \cdots, x_{n-1}, x_n = b$ such that $\Delta_1 x = x_1 - a$, $\Delta_2 x = x_2 - x_1$, $\cdots$, $\Delta_n x = b - x_{n-1}$, each interval being numerically less than or equal to δ. Let $\xi_1, \xi_2, \cdots, \xi_n$ be any points of the intervals x_0x_1, $x_1x_2, \cdots, x_{n-1}x_n$ respectively and let

$$S_\delta = f(\xi_1)\Delta_1 x + f(\xi_2)\Delta_2 x + \cdots + f(\xi_n)\Delta_n x = \sum_{k=1}^{n} f(\xi_k)\Delta_k x.$$

If the many valued function S_δ of δ approaches a single limiting value as δ approaches zero, then

$$\lim_{\delta \doteq 0} S_\delta = \int_a^b f(x)dx.$$

$S_a^b\delta$ represents the sum S_δ over the interval ab.

LEMMA I. *The function S_δ in the definition of the definite integral approaches its limit uniformly with respect to x, i. e., for every ϵ there exists a δ_ϵ such that*

$$\left| \int_{x_1}^{x_2} f'(x)dx - S_{x_1}^{x_2}\delta_\epsilon \right| < \epsilon$$

for every $S_{x_1}^{x_2}\delta_\epsilon$ and for every pair of points x_1 and x_2 on ab.

Proof: Since by hypothesis

$$\lim_{\delta \doteq 0} S_a^b\delta = \int_a^b f(x)dx,$$

here exists for a given ϵ a $\delta_{\epsilon/2}$ such that

$$\left| \int_a^b f(x)dx - S_a^b\delta_{\epsilon/2} \right| < \frac{\epsilon}{2} \tag{1}$$

for every $S_a^b\delta_{\epsilon/2}$. Then for any pair of points x_1, x_2 on ab

$$\left| \int_a^{x_1} f(x)dx - S_a^{x_1}\delta_{\epsilon/2} + \int_{x_1}^{x_2} f(x)dx - S_{x_1}^{x_2}\delta_{\epsilon/2} + \int_{x_2}^b f(x)dx - S_{x_2}^b\delta_{\epsilon/2} \right| < \frac{\epsilon}{2}. \tag{2}$$

Since $f(x)$ is integrable on ax_1 and x_2b, it follows that there exists a $\delta'_{\epsilon/2}$ for ax_1 and a $\delta''_{\epsilon/2}$ for x_2b such that

$$\left| \int_a^{x_1} f(x)dx - S_a^{x_1}\delta'_{\epsilon/2} + \int_{x_2}^b f(x)dx - S_{x_2}^b\delta'_{\epsilon/2} \right| < \epsilon/2. \tag{3}$$

Then from (2) and (3)

$$\left| \int_{x_1}^{x_2} f(x)dx - S_{x_1}^{x_2}\delta_{\epsilon/2} \right| < \epsilon.$$

Hence $\delta_{\epsilon/2}$ is the δ_ϵ required by the lemma.*

Definition: The difference between the least upper and the greatest lower bound of a function on an interval is the oscillation of the function on that interval.

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*If $\delta_{\epsilon/2}$ is so chosen that $\delta_{\epsilon/2} \leqq \delta'_{\epsilon/2}$ and $\delta_{\epsilon/2} \leqq \delta''_{\epsilon/2}$, the argument is a little more obvious, though this restriction is not necessary.

LEMMA II. *If the oscillation of a function on an interval ab is less than $\epsilon/2$ then for any two partitions π and π' of ab*

$$\left|\sum_{i=1}^{n}|\Delta_i x|\cdot|f(\xi_i)|-|\sum_{i=1}^{n'}\Delta_i' x f(\xi_i')|\right|<\epsilon|b-a|$$

when n and n' are the numbers of intervals on the partitions π and π' respectively.

Proof: Let B be any value of $f(x)$ on $a\,b$. Then for any interval $x_i x_{i+1}$ of the partition π

$$\left||f(\xi_i)|\cdot|\Delta_i x|-|\beta|\cdot|\Delta_i x|\right|<\frac{\epsilon}{2}\cdot\Delta_i x.$$

Hence

$$(1)\qquad \left|\sum_{i=1}^{n}|\Delta_i x|\cdot|f(\xi_i)|-|\beta|\cdot|b-a|\right|<\frac{\epsilon}{2}|b-a|.$$

Similarly for any interval $x_i' x_{i+1}'$ of the partition π'

$$|f(\xi_i')\,\Delta_i' x-\beta\cdot\Delta_i' x|<\frac{\epsilon}{2}|\Delta_i' x|.$$

Hence

$$\left|\sum_{i=1}^{n'}f(\xi_i')\Delta_i' x-\beta\cdot(b-a)\right|<\frac{\epsilon}{2}|b-a|$$

and

$$(2)\qquad \left||\sum_{i=1}^{n'}f(\xi_i')\Delta_i' x|-|\beta|\cdot|b-a|\right|<\frac{\epsilon}{2}|b-a|.$$

From (1) and (2) we have

$$\left|\sum_{i=1}^{n}|\Delta_i x|\,|f(\xi_i)|-|\sum_{i=1}^{n'}\Delta_i' x\cdot f(\xi_i')|\right|<\epsilon|b-a|.$$

THEOREM II. *If $\int_a^b f(x)dx$ exists, then $V_a^b\int_a^x f(x)dx$ exists and*

$$V_a^b\int_a^x f(x)dx=\int_a^b|f(x)|\,dx.$$

Proof: Since for any two values of x, x_1 and x_2 on ab

$$(1)\qquad \left|\int_{x_1}^{x_2}f(x)dx\right|\leqq\int_{x_1}^{x_2}|f(x)|dx$$

and since by a well-known theorem

$$\int_{x_1}^{x_2} |f(x)|\,dx$$

exists if

$$\int_{x_1}^{x_2} f(x)dx$$

exists, it follows from (1) and Theorem I that

$$V_a^b \int_a^x f(x)dx$$

exists and that

$$V_a^b \int_a^x f(x)dx \leqq V_a^b \int_a^x |f(x)|\,dx. \tag{2}$$

It remains to show that the sign $<$ in (2) is impossible. Since $f(x)$ is integrable on ab it follows by a well known theorem that for any pair of positive numbers σ and λ there exists a partition π such that the sum of the intervals on which the oscillation of $f(x)$ is greater than σ is less than λ.

For a preassigned ϵ let $\lambda = \epsilon/M$, where M is the difference between the least upper and the greatest lower bound of $f(x)$ on ab. Let $\sigma = \epsilon/4|b-a|$. Denote by I the set of intervals on ab on which the oscillation is less than σ and by I' the complementary set of intervals. Let n be the number of intervals in I. By Lemma I there exists a δ such that

$$\left| \int_{x_1}^{x_2} f(x)dx - S_{x_1}^{x_2}\delta \right| < \frac{\epsilon}{2n} \tag{3}$$

for every pair of points x_1 and x_2 on ab and for every $S_{x_1}^{x_2}\delta$. Let x_1, x_2 be the extremities of a segment of the set I. By Lemma II

$$\left| |S_{x_1}^{x_2}\delta| - |x_1 - x_2| \cdot |f(\xi_2)| \right| < \frac{\epsilon}{2|b-a|}\ |x_1 - x_2|. \tag{4}$$

Denoting the integrals on the segments of the set I by

$$\int_i f(x)dx,$$

the lengths of the segments by Δ_i, and the sums S_δ on the segments of I by $S_{x_j}^{x_{j+1}}$, we have from (4)

$$\left| \sum_{j=1}^{j=2n-1} |S_{x_j}^{x_{j+1}}\delta| - \sum_{i=1}^{n} \Delta_i |f(\xi_i)| \right| < \frac{\epsilon}{2}$$

where $2n-1$ is the number of extremities of the segments of I and where j takes only odd values.

Then from (3)

$$\left| \sum_{j=1}^{2n-1} |S_{x_j}^{x_{j+1}}\delta| - \sum_{i=1}^{n} \left| \int_i f(x)dx \right| \right| < \frac{\epsilon}{2},$$

since if

$$\left| \int_{x_1}^{x_2} f(x)dx - S_{x_1}^{x_2}\delta \right| < \frac{\epsilon}{2n},$$

then

$$\left| \left| \int_{x_1}^{x_2} f(x)dx \right| - |S_{x_1}^{x_2}\delta| \right| < \frac{\epsilon}{2n}.$$

Hence

$$\left| \sum_{i=1}^{n} \Delta_i |f(\xi_i)| - \sum_{i=1}^{n} \left| \int_i f(x)dx \right| \right| < \epsilon.$$

Since

$$\lim_{\delta=0} \sum_{j=1}^{n'} \Delta_j |f(\xi_j)| = \int_i |f(x)|dx$$

and since ϵ is arbitrary it is evident that the least upper bound of

$$\sum_{j=1}^{n'} \left| \int_j f(x)dx \right|$$

for the set of all partitions of I cannot be less than

$$\sum_{i=1}^{n} \int_i |f(x)|dx$$

(where

$$\int_j f(x)dx$$

denotes the integral on an interval formed by partitioning an interval of I.)

Since the sum of the lengths of the interval on which the oscillation is greater than σ is less than $\lambda = \epsilon/M$ and since by a known theorem

$$\sum_{i=1}^{n''} \int_{I'} |f(x)|dx < \lambda \cdot M = \epsilon$$

(where

$$\int_{I'} f|(x)dx$$

denotes the integral on a segment of I') it follows from the arbitrary character of ϵ that the least upper bound of

$$\left[\sum_{i=0}^{n} \left| \int_{x_i}^{x_{i+1}} f(x)dx \right|\right]$$

for the set of all partitions of $a\,b$ cannot be less than

$$\int_a^b |f(x)|\,dx,$$

which proves the theorem.

Definition: The length of a curve represented by the equation $y = f(x)$ on an interval $a\,b$ is the least upper bound of the set of sums

$$\left[\sum_{i=1}^{n} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}\right]$$

for the set of all partitions $a = x_1, x_2, \cdots, x_{n-1}, x_n = b$ of the interval ab, where y_i corresponds to x_i by the functional correspondence $y = f(x)$.

THEOREM III. *If on the interval ab and on every subinterval of ab the functions $f_1(x)$ and $f_2(x)$ are of equal variation then the curves represented by $y = f_1(x)$, and $y = f_2(x)$ are of equal length on the interval ab.*

Proof: Consider any partition π of ab consisting of the points $x_i (i = 0, \cdots, n)$. Then

$$\sum_{i=0}^{n} \sqrt{(x_i - x_{i+1})^2 + (f(x_i) - f(x_{i+1}))^2}$$

is one sum of the set of sums whose least upper bound is the length of the curve $y = f(x)$ on ab.

By hypothesis $f_1(x)$ and $f_2(x)$ are of the same variation on any interval $x_i x_{i+1}$ of π. Obviously the variation of $f_1(x)$ on $x_i\ x_{i+1}$ is equal to or greater than $|f_1(x_i) - f(x_{i+1})|$. Let ϵ be any preassigned positive number and let $\epsilon' = \epsilon/n$ where n is the number of interval in the partition π. Then there is a partition π'_i of $x_i\ x_{i+1}$ consisting of the points $x'_0 = x_i, x'_2, \cdots, x'_j, \cdots, x'_{n'} = x_{i+1}$ such that

$$(1) \qquad |f_1(x_i) - f_1(x_{i+1})| - \sum_{j=0}^{n'} |f_2(x_j) - f_2(x_{j+1})| < \epsilon'.$$

By means of a broken line whose segments are equal in length to the lengths of the segments connecting the points $(x_j, f_2(x_j))$ and $(x_{j+1}, f_2(x_{j+1}))$ connect the point $(x_i, f_2(x_i))$ with a point (x_{i+1}, y_1) on the line $x = x_{i+1}$ in such manner that the slopes of these segments are, say, all positive. Then by (1)

$$(2) \qquad |f_1(x_i) - f_1(x_{i+1})| - |f_2(x_1) - y_1| < \epsilon'.$$

Hence obviously

$$(3) \quad \sqrt{(x_i - x_{i+1})^2 + [f_1(x_i) - f_1(x_{i+1})]^2} - \sum_{j=0}^{n'} \sqrt{(x_j - x_{j+1})^2 + [f_2(x_j) - f_2(x_{j+1})]^2} < \epsilon' \cdots.$$

Let π'', consisting of the points $x_1 = a, x_2, \cdots, x_k, \cdots, x_{n''} = b$, be the partition of ab containing all the points of the partitions π'_i of the intervals $x_i x_{i+1}$, and let n'' be the number of intervals in this partition. Then from (3)

$$(4) \quad \sum_{i=0}^{n} \sqrt{(x_i - x_{i+1})^2 + [f_1(x_i) - f_1(x_{i+1})]^2} - \sum_{k=0}^{n''} \sqrt{(x_k - x_{k+1})^2 + [f_2(x_k) - f_2(x_{k+1})]^2} < \epsilon' n = \epsilon \cdots.$$

By a well known theorem the lengths on ab of the curves $y = f_1(x)$ and $y = f_2(x)$ both exist, since the functions are of limited variation. Moreover, it follows from (4) that the length of the curve $y = f_2(x)$ cannot be less than that of $y = f_1(x)$, ϵ being arbitrary. In the same manner it is shown that the length of the curve $y = f_1(x)$ cannot be less than the length of the curve $y = f_2(x)$. Hence the curves are of equal length and the theorem is proved.

Theorems II and III yield the following interesting

COROLLARY. *If $f(x)$ is such that*

$$\int_a^x f'(x)dx = f(x) - f(a)$$

on ab then the length on ab of the curve $y = f(x)$ can be found by finding the length of the curve

$$y = \int_a^x |f'(x)|\, dx$$

on this interval.

Proof: By Theorem II the curves

$$y = \int_a^x f'(x)dx = f(x) - f(a) \quad \text{and} \quad y = \int_a^x |f'(x)|\, dx$$

have the same variation on ab and on every subinterval of ab, whence by theorem III their lengths are equal.

This corollary reduces the problem of finding the length of a curve of the class specified in the corollary to the finding of the length of a non-oscillating curve.

We now use theorem II to give a fresh proof of a theorem on improper definite integrals.

Definition: A function is integrable at a point x_0 if there exists an interval containing x_0 as an interior point on which $f(x)$ is properly integrable. (A function is properly integrable on an interval only in case it is bounded on that interval.)

Consider a function $f(x)$ which is integrable at every point of the interval ab except at a set of points $[P]$ which is of content zero.*

Let $[I]$ be any finite set of non-overlapping intervals on ab such that no point of $[P]$ lies on an interval of $[I]$. Then the integral of $f(x)$ exists properly on every interval of $[I]$. Denote by

$$\int_{aI}^{b} f(x)dx$$

the sum of the integrals of $f(x)$ on the intervals of $[I]$.

* A set of points $[P]$ is of content zero if for every positive number ϵ there exists a finite set of intervals $[I']$ such that every point of $[P]$ lies on at least one interval of $[I']$ and further such that the sum of the lengths of the intervals of $[I']$ is less than ϵ.

Denote by $m[I]$ the sum of the lengths of these intervals and let $D = |a - b|$.

If the limit

$$\lim_{m[I]=D} \int_{aI}^{b} f(x)dx$$

exists and is finite, this limit is said to be the improper definite integral of $f(x)$ on ab.

THEOREM IV. *The improper definite integral regarded as a function of the upper limit of integration is of limited variation on any interval ab.*

Proof: Since by hypothesis the improper definite integral of $f(x)$ exists on ab it follows that there exists an M such that for every set of interval $[I]$

$$\left| \int_{aI}^{b} f(x)dx \right| < M. \tag{1}$$

If the theorem fails to hold, *i. e.*, if for every M' there is a partition π of ab such that

$$\sum_{i=0}^{n} \left| \int_{x_i}^{x_{i+1}} f(x)dx \right| > M', \tag{2}$$

then for a certain subset $x_j x_{j+1}$ of the set of interval $x_i x_{i+1}$

$$\sum_{j=0}^{n'} \left| \int_{x_j}^{x_{j+1}} f(x)dx \right| > \frac{M'}{2}. \tag{3}$$

(The integrals in (3) are improper definite integrals.) Since the improper definite integral exists on each of the intervals $x_j x_{j+1}$ there exists a set of segments $[I']$ such that

$$\int_{aI'}^{b} f(x)dx > \frac{M'}{2}. \tag{4}$$

If $M' = 2M$ then (4) contradicts (1). Hence the theorem is proved.

THEOREM V. *If the improper definite integral of $f(x)$ exists on the interval ab then the improper definite integral of $|f(x)|$ exists on ab.*

Proof: By theorem IV

$$\int_a^x f(x)dx$$

is of limited variation on ab. Hence there exists a number M such that for every set of intervals $[I]$

$$V_a^b \int_{x_i}^{x_{i'}} f(x)dx < M.$$

But by Theorem II,

$$V_a^b \int_{x_i}^{x_{i'}} f(x)dx = \int_{aI}^{bI} |f(x)|\,dx < M,$$

which proves the theorem.*

CHICAGO,
July 17, 1906.

A NOTE ON TRANSITIVE GROUPS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 3, 1906.)

THREE unconnected topics in the theory of transitive substitution groups are touched on in this note.

1.

THEOREM I. *The largest subgroup of a transitive group G of degree n, in which a subgroup H leaving fixed m $(0 < m < n)$ letters is invariant, has as many transitive constituents in these m letters as there are different conjugate sets in G_1 (a subgroup of G, that leaves one of the m letters fixed) which, under the substitutions of G, enter into the complete set of conjugates to which H belongs. Moreover, the degree of each of these constituents is proportional*

* For a general discussion of the improper definite integral see E. H. Moore, "Concerning Harnack's theory of improper definite integrals." *Transactions Amer. Math. Society*, volume 2, pp. 296–330, and pp 459–475 same volume. See also references given in this paper by Professor Moore. For a proof of Theorem V. of the present note see Jordan, Cours d'analyse, ed. 2, vol. 2 (1894), pp. 46 ff.

*to the number of subgroups in the several conjugate sets of G_1 in question.**

Let g be the order of G, and let those conjugates of H, which are found in G_1, lie in κ different sets, in so far as they are permuted by the substitutions of G_1 only, with c conjugates in the set including H, c_1 in a second set, and so on. In G_1, H is invariant in a subgroup of order g/nc, while H_1, a subgroup in the second set, is invariant in a group of order g/nc_1. The largest subgroup I of G in which H is invariant is of order $gm/n\sigma$ ($\sigma = c + c_1 + \cdots + c_{\kappa-1}$). Now I does not connect transitively the $n - m$ letters displaced by H and the m letters it leaves fixed. Since the largest subgroup of G_1 in which H is invariant is of order g/nc, I has one transitive constituent of degree $(gm/n\sigma) \div (g/nc) = mc/\sigma$ in letters left fixed by H. The group H_1 is transformed into H by some substitution S not in G_1. Then in the subgroup $G_2 = S^{-1}G_1S$, H belongs to a set of g/nc_1 conjugates (conjugate under the substitutions of G_2). Hence I has a transitive constituent of degree mc_1/σ in letters left fixed by H. Continuing thus, we see that I has transitive constituents of degrees mc/σ, $mc_1/\sigma, \cdots, mc_{\kappa-1}/\sigma$. These constituents involve all the letters left fixed by H. This completes the proof of the theorem.

COROLLARY. *If H is a Sylow subgroup of G_1, I has a transitive constituent of degree m.*

2.

THEOREM II. *If the class of a t-ply transitive group ($t > 3$) is not less than $n - 2t + 5$, then the degree n is less than or equal to $\frac{1}{2}(t^2 - t + 2) + (t - 1)!$*

There are three cases to consider separately:

G contains a substitution of order 2 and (1) of degree $n - t + 2$, (2) of degree $n - t + 1$, (3) of degree $n - t - \epsilon$ ($0 \leqq \epsilon \leqq t - 5$). Since the subgroup of G that leaves fixed $t - 2$ letters is of even order, it always contains a substitution of order 2 falling under one of the three following categories:

1. If there is one substitution of order 2 on just $n - t + 2$ letters in G, there are at least $[n(n-1)\cdots(n-t+3)(n-t+1)]/(t-2)!$ The total number of transpositions involved in these substitutions is $n(n-1)\cdots(n-t+1)/2(t-2)!$ Of these substitutions at least $n(n-1)\cdots(n-t+1)/n(n-1)$

* Cf. Burnside, Theory of Groups (1897), p. 202, where a special case is proved.

$(t-2)!$ have some particular transposition as (ab) in common. Again by the same reasoning at least $n(n-1)\cdots(n-t+1)/n(n-1)(n-2)(n-3)(t-2)!$ that have (ab) in common have also a transposition (cd) in common. Finally the number of these substitutions which have $t-2$ common transpositions is at least

$$(1) \qquad \frac{n(n-1)\cdots(n-t)(n-t-2)\cdots(n-3t+8)}{(t-2)!\ n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq 1.$$

That this number cannot be greater than unity comes from the fact that G is by hypothesis of class $n-2t+5$, while the product of two substitutions with $t-2$ transpositions in common is of degree less than or equal to $n-2t+4$. This inequality is of the first degree in n, so that we obtain from it

$$(2) \qquad n \leqq \tfrac{1}{2}(t^2-3t+4)+(t-2)!.$$

2. In case G has a substitution of order 2 and degree $n-t+1$, we get by the same process

$$(3) \qquad \frac{n(n-1)\cdots(n-t+1)(n-t-1)\cdots(n-3t+7)}{(t-1)!\,n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq 1,$$

from which

$$(4) \qquad n \leqq \tfrac{1}{2}(t-t-2)+(t-1)!$$

When $t=4$, G is of class $n-3$, and the above inequality gives $n \leqq 11$.

3. The inequality in this case is

$$(5) \qquad \frac{n(n-1)\cdots(n-t+1)(n-t-\varepsilon)(n-t-\varepsilon-2)\cdots(n-3t-\varepsilon+6)}{(t+\varepsilon)(t+\varepsilon-1)\cdots(\varepsilon+1)(n)(n-1)\cdots(n-2t+3)} \leqq 1.$$

It is of the second degree in n, and gives

$$(6) \qquad n \leqq \tfrac{1}{2}(t^2-3t-8+2t\varepsilon-4\varepsilon)+\sqrt{(t+\varepsilon)!/\varepsilon!}$$

The limit $\frac{1}{2}(t-t+2)+(t-1)!$ is always the highest of the three.

In this connection it is easy to establish the following

THEOREM III. *If the subgroup leaving t letters fixed in a t-ply $(t>2)$ transitive group G is of even order, the degree of*

G is less than $\frac{1}{2}(t^2-t+6)+\varepsilon(t-1)+(t+\varepsilon)!/\varepsilon!$ unless the class is less than $n-2t+3$.

The inequality

$$\frac{n(n-1)\cdots(n-t+1)(n-t-\varepsilon)(n-t-\varepsilon-2)\cdots(n-3t-\varepsilon+4)}{n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq \frac{(t+\varepsilon)!}{\varepsilon!}$$

where $0 \leqq \varepsilon \leqq t-3$, is found as before, and from it the theorem follows.

3.

Another theorem of value in the applications is the following:

THEOREM IV. *A doubly transitive group cannot contain an invariant imprimitive subgroup unless its degree is a power of a prime. Then the group is a subgroup of the holomorph of the abelian group of order p^α and type* $(1, 1, \cdots)$.

On pages 193 and 194 of his Theory of Groups Burnside proves that the invariant imprimitive subgroup H is of degree n and class $n-1$ and that the $n-1$ substitutions of degree n in H form a single conjugate set under G. Then by Frobenius's theorem on groups of "class $n-1$," H contains a characteristic subgroup of degree and order n which is abelian with all its operators of the same order.

STANFORD UNIVERSITY,
June 19, 1906.

DIFFERENTIAL GEOMETRY OF n DIMENSIONAL SPACE.

Sur les Systèmes Triplement Indéterminés et sur les Systèmes Triple-Orthogonaux. Par C. GUICHARD. Scientia, no. 25. Gauthier-Villars, Paris, 1905. viii + 95 pp.

DURING the past ten years the field of differential geometry has been greatly enriched by the researches of M. Guichard. The eminent geometer has made a profound study of the properties of ordinary space by means of operations defined for space of n dimensions. He has introduced two elements depending upon two variables; they are the *reseau* and the *congruence*. By definition, a point of space in n dimensions

describes a *reseau*, if its n rectangular coordinates are solutions of an equation

$$\frac{\partial^2\theta}{\partial u\partial v} = P\frac{\partial\theta}{\partial u} + Q\frac{\partial\theta}{\partial v}.$$

In the customary terminology of differential geometry of ordinary space a reseau is evidently a conjugate system of curves upon the surface locus of these curves. A *congruence* is a doubly infinite system of straight lines which touch two series of curves, as in ordinary space. As thus defined, the lines of the congruence form two families of developable surfaces, a family being given when one of the variables is constant. It is not our purpose to discuss the beautiful theory which Guichard has developed by means of these two elements and set forth in two memoirs, "Les systèmes orthogonaux et les systèmes cycliques" (*Annales de l'Ecole Normale*, 1897, 1898, 1903), but merely to make this reference to it. For it is the application and extension of these methods to the case of triply indeterminate forms which Guichard has developed in the book which is before us for review. This book is more of the nature of a syllabus than a treatise and the matter is so essentially new that it seems advisable to give an outline of its contents rather than attempt a critical review.

In the present theory Guichard makes use of three elements: point systems, line systems and plane systems. The coördinates of a point M in space of n dimensions being given in terms of three parameters u_1, u_2, u_3, M describes a point system if, one of the parameters remaining fixed, the doubly indeterminate system described by M is a reseau. In ordinary space a point system is therefore a triply conjugate system. A line system is formed by the tangents to any one of the curves described by M when only one of the parameters varies. A plane system is described by the plane of two tangents of a point system.

If MT_1 denotes the tangent to the curve of parameter u_1 of a point system M, there are upon MT_1 two determinate points, say A and B, which generate point systems and for which MT_1 is tangent to the curves of parameters u_2 and u_3 respectively. Thus from a given point system we get six other point systems without quadrature, so that we have a transformation of point systems. Moreover, these new point systems are related among themselves in such a way, that if we go, for instance, from M along MT_1 to A, and then along the tangent at A to the curve

of parameter u_3, to the point F generating the point system for which AF is tangent at F to the curve of parameter u_1, the line MF is tangent to the curve of parameter u_2 at F. Darboux considered these relations and called them transformations of Laplace (Leçons, IV^e Parte, Chapter XII), as they are generalizations of the transformations of Laplace in the theory of conjugate systems. We have remarked that these transformations form cycles of three. This fact is important as showing that a plane of two tangents to a point system bears the same relation to two other point systems. Consequently, a plane which generates a plane system has three focal points, each of which generates a point system. The lines joining them are the focal lines of the plane system; and the three points, as M, A, B, on a line generating a line system are called its focal points. These definitions and the derivation of the properties of the defined quantities constitute the subject matter of Chapters I and II. The latter contains also the definition of parallel systems. Two point systems are said to be parallel when their corresponding tangents are parallel. Similar definitions are given of the parallelism of line and plane systems. The determination of parallel systems from a given one requires integration. In this connection the following fundamental theorem obtains: if two systems are parallel, the corresponding focal systems are parallel.

With each system, whether point, plane or line, there are associated systems of the other two kinds in the following ways. A line system and a point system are said to be assembled, when the point M lies on the corresponding line Δ and as M describes the curve of parameter u_i the line Δ envelops a curve of parameter u_i at the corresponding focal point. A point system and a plane system are assembled when the foci of the plane are situated on the tangents to the corresponding curves of the point system. And a line system and a plane system are assembled when the generating line of the former lies in the generating plane of the latter, the foci of the former lying on the corresponding focal lines of the latter. A fundamental theorem is that when two systems are assembled, their focal elements are assembled. For instance, if a point and line system are assembled, every focal plane of the point is assembled to a focal point of the line. There is also the following law of parallelism: if two systems are parallel, every system assembled with the one is parallel to a system assembled with the other.

In ordinary three dimensional geometry, if we have a triply conjugate system of surfaces, the normals to these surfaces are parallel to the directions of the curves of intersection of other triply conjugate systems, and the determination of the latter requires the integration of a system of partial differential equations of the first order. In Chapter IV, Guichard has generalized this result for space of n dimensions and gathered together his results in a law of orthogonality of elements. This law varies with the order of the space in the following way:

"In spaces of order $3p$, this law makes correspond to a point system, a point system; to a line system, a plane system and inversely. In spaces of order $3p + 1$, it makes correspond to a plane system, a plane system; to a line system, a plane system and inversely. In spaces of order $3p + 2$, it makes correspond to a line system, a line system; to a point system, a plane system and inversely." For spaces of order $3p$ the following theorems obtain:

"If two point systems are orthogonal, every focal line of the one is orthogonal to the focal plane of the same kind of the other. Every line system assembled with the one is orthogonal to a plane system assembled with the other and inversely."

"If a line system and a plane system are orthogonal, every focal point of the line is orthogonal to the focal point of the same kind of the plane; every point system assembled with the line is orthogonal to a point system assembled with the plane and inversely, etc." Similar theorems exist for spaces of order $3p + 1$ and $3p + 2$.

Beginning with Chapter V, the discussion of point systems is concerned chiefly with those for which the tangents to the parametric curves are perpendicular to one another. When these tangents are not isotropic, the system is called a system O; otherwise, a singular rectangular system. The theory of these systems is intimately related to the theory of a determinant Δ whose elements are the coefficients of an orthogonal substitution in space of n dimensions, the elements of the last three rows being the direction cosines of the tangents to the system O and the elements of the other rows being such that their first derivatives are very simple linear functions of the elements of the same columns. The coefficients in these linear functions are called the rotations of the determinant; they must satisfy a system of partial differential equations of the first order. These equations are such that the last three rows of the

determinant Δ determine it completely. Consequently to each system O there corresponds a unique determinant, and to a given determinant there corresponds a parallel system of systems O. There are systems O in spaces of any order. The determinant associated with a system O gives a means of finding certain systems assembled with O. When $n > 3$, there are ∞^{n-4} lines, passing through the point M of a system O perpendicular to the three tangents, each of which describes a line system. When $n > 4$ there can be normal line systems generated by isotropic lines. The first kind are called ordinary normal systems; the latter isotropic.

In Chapter VI the notion of the projection of a system from one space into another is introduced. This leads to a generalization of the various systems. Thus, a point M of space of n dimensions describes a system p, O when it can be considered as the projection of a system O of space of $n + p - 1$ dimensions; that is, if $x_1, x_2, \cdots, x_n$ are the coördinates of M, the equations of Laplace which are satisfied by these coördinates admit $p - 1$ other solutions $y_1, \cdots, y_{p-1}$ such that

$$\Sigma dx^2 + \Sigma dy^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2,$$

the functions h_1, h_2, h_3 being different from zero; the quantities $y_1, \cdots, y_{p-1}$ are called the complementary coördinates of the system. A system of lines D is called I, if the sum of the squares of the direction parameters $(X_1, \cdots, X_n)$ of the line D is zero. The isotropic systems of normals to a system O, previously referred to, are of this kind and they are the only ones. When a line system is such that the equations of Laplace satisfied by the quantities X admits $p - 1$ other solutions $(Y_1, \cdots, Y_{p-1})$ such that $\Sigma X^2 + \Sigma Y^2 = 0$, the system is said to be p, I. It is shown that all the ordinary normals to a system O form systems $2I$ and that all other assembled line systems are $3I$. A plane system assembled with a system O is said to be Ω; and it is p, Ω if it can be considered as the projection of a system Ω in space of order $n + p - 1$. Further, it is shown that among the plane systems assembled with a point system $2O$, one series of parallel plane systems are Ω and the others are 2Ω. Thus, there are the following assembled systems for space of three dimensions:

Assembled Systems.

Point.	Plane.	Lines.
O	Ω	$3I$
$2,O$	$\begin{cases} 1 & \Omega \\ \text{Others} & 2\Omega \end{cases}$	$\begin{cases} 1 & 3I \\ \text{Others} & 4I \end{cases}$
$3,O$	$\begin{cases} 2 & \Omega \\ \infty^1 & 2\Omega \\ \text{Others} & 3\Omega \end{cases}$	$\begin{cases} 2 & 3I \\ \infty^1 & 4I \\ \text{Others} & 5I \end{cases}$
p,O	$\begin{cases} \infty^{p-3} & p-2, \Omega \\ \infty^{p-2} & p-1, \Omega \\ \text{Others} & p, \Omega \end{cases}$	$\begin{cases} \infty^{p-3} & p-2, I \\ \infty^{p-2} & p-1, I \\ \text{Others} & p, I \end{cases}$

There are other assembled systems for three dimensional space, which for lack of space we cannot indicate, and similar tables of assembled systems for spaces of four, five and six dimensions also are given.

In Chapter VII it is shown how the preceding theories enable one to find new triply orthogonal systems. They arise as a result of the search for point, plane and line systems which present two properties; for instance, a point system which is p, O and q, O. In this connection it is found to be advantageous to consider pairs of point systems $M(x_1, \cdots, x_n)$ and $N(y_1 \cdots, y_m)$ which have the property that the Laplace equations are the same for both systems. It results that

$$\Sigma dx^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2,$$

$$\Sigma dy^2 = h_1'^2 du_1^2 + h_2'^2 du_2^2 + h_3'^2 du_3^2,$$

where

$$h_1' = h_1 U_1, \quad h_2' = h_2 U_2, \quad h_3' = h_3 U_3,$$

U_1, U_2, U_3 being functions of u_1, u_2, u_3 respectively. The two systems are said to be associate. Evidently the point whose coördinates are $x_1, x_2, \cdots, y_1, \cdots, y_m$ describes a system O in space of $m + n$ dimensions. Hence a system O associate to a system O in a space of m dimensions is a system $O, (m + 1)\ O$ and inversely. From this it is seen that the systems O, $4O$ are the simplest. By considering systems assembled to a system O, $4O$ in space of three dimensions it is found that the determination of systems $3I$, $4I$; Ω, 2Ω; $2O$, $3O$ are equivalent problems to the determination of systems O, $4O$. Similar and more general results follow from the discussion of systems O, $4O$ in spaces of four, five and six dimensions. And it is found that all problems of finding systems possessing two properties re-

duces to the search for associate systems O. Interesting particular problems are furnished by considering the cases where certain of the above functions U_1, U_2, U_3 are constants.

As an example of the theory, the determination of systems O, $4O$ in space of three dimensions is treated at length in Chapter VIII and also the analytical expressions for transformations to the assembled systems.

Two point systems are applicable when

$$\Sigma dx^2 = \Sigma dy^2.$$

In the last chapter Guichard considers systems O in three dimensions which are applicable to systems O in six dimensions. It is a study of the determinants of these systems, and from the form of the equations satisfied by the rotations it is found that the problem reduces to the determination of two triply orthogonal systems in space of three dimensions for which the rotations β_{ik} and β'_{ik} of the respective systems satisfy the conditions

$$\beta'_{ik} = \beta_{ki}.$$

The rotations β_{ik} are those defined by Darboux in his Leçons. Of this kind are the systems of Ribaucour for which all the surfaces in one family have the same spherical representation.

We have noticed a few typographical errors which, however, are in most cases errors in reference to sections and not serious. The form of the book is attractive and we should gladly welcome the publication in similar form of the two memoirs on the geometry of two indeterminates, previously mentioned. One who reads these works of Guichard looks forward to the time when he can read them again and try out the ideas which suggest themselves in the reading.

L. P. EISENHART.

SHORTER NOTICES.

Bibliography of Quaternions and Allied Systems of Mathematics. By ALEXANDER MACFARLANE. University Press, Dublin, 1904.

THE International Catalogue of Scientific Literature places the following branches of mathematics under the head of universal algebra: (1) calculus of operations. (2) general theory of complex numbers. (3) quaternions. (4) Ausdehnungslehre and vector analysis. (5) matrices. (6) other special sorts of complex numbers. (7) algebra of logic.

The work under review contains a bibliography of topics 2, 3, 4, and 5. Titles in which vector ideas and methods are applied are also included. Although the subject of bilinear forms is not mentioned in the International Catalogue, Professor Macfarlane very properly includes certain titles in this field which deal with the theory of matrices. It is an extremely difficult matter to prepare a complete bibliography for a field as wide as Professor Macfarlane has chosen, and his approximation will be keenly appreciated, especially by those to whom large libraries are not accessible. The fields of quaternions and vector analysis seem to be carefully covered. In the allied branches of general theory of complex numbers and matrices the bibliography is not so complete, though most of the fundamental memoirs are noted. A partial list of omissions is appended, which contains among others important titles by Weierstrass, Frobenius, Molien and Dedekind.

CARTAN.

1897. "Sur les systèmes de nombres complexes." *Comptes Rendus*, page 1217.

1897. "Sur les systèmes réels de nombres complexes." *Comptes Rendus*, page 1296.

1898. "Les groupes bilinéaires et les systèmes de nombres complexes." *Annales de la Faculté des Sciences de Toulouse*, volume 12, page 81.

DEDEKIND.

1885. "Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen." *Göttinger Nachrichten*, page 141.

1887. "Erläuterungen zur Theorie der sogenannten allgemeinen complexen Grössen." *Göttinger Nachrichten*, page 1.

FROBENIUS.

1896. "Ueber die cogredienten Transformationen der bilinearen Formen." *Sitzungsberichte der Berliner Akademie*, page 7.

HAUSDORFF.

1900. "Zur Theorie der Systeme complexer Zahlen." *Leipziger Berichte*, page 43.

HAWKES.

1902. "Estimate of Peirce's Linear Associative Algebra." *American Journal of Mathematics*, volume 24, page 87.
"On hypercomplex numbers." *Transactions of the American Mathematical Society*, volume 3, page 312.
"Enumeration of non-quaternion number systems." *Mathematische Annalen*, volume 58, page 361.

HILBERT.

1896. "Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen." *Göttinger Nachrichten*, page 179.

JOLY.

1898. "The associative algebra applicable to hyperspace." *Proceedings of the Irish Academy.*

MOLIEN.

1892. "Ueber Systeme höherer complexer Zahlen." *Mathematische Annalen*, volume 41, page 83.
1893. "Berichtigung zu dem Aufsatze: Ueber Systeme höherer complexer Zahlen." *Mathematische Annalen*, volume 42, page 308.

PIERCE.

1903. "Classification and properties of dual conical congruences." Dissertation, Zürich.

STUDY.

1890. "Ueber Systeme complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen." *Monatshefte für Mathematik*, page 283.

1898. "Beweis eines von Herrn Dedekind angegebenen Satzes." *Göttinger Nachrichten*, page 1.

TABER.

1892. "On a theorem of Sylvester's relating to non-degenerate matrices." *Proceedings of the American Academy*, page 46.

"Note on the representation of orthogonal matrices." *Proceedings of the American Academy*, page 163.

1893. "On real orthogonal substitutions." *Proceedings of the American Academy*, page 212.

1894. "On the automorphic linear transformation of bilinear forms." *Proceedings of the American Academy*, page 178.

"On orthogonal substitutions." *Bulletin of the New York Mathematical Society*, volume 3, page 251.

1897. "Notes on the theory of bilinear forms." *Bulletin of the American Mathematical Society*, volume 3, page 156.

WEIERSTRASS.

1884. "Zur Theorie der aus n Haupteinheiten gebildeten complexen Zahlen." *Göttinger Nachrichten*, page 395.

H. E. HAWKES.

An Elementary Text-book on the Differential and Integral Calculus. By WILLIAM H. ECHOLS. Professor of Mathematics in the University of Virginia. New York, Henry Holt and Company, 1902.

IN this book Professor Echols has made a commendable and, on the whole, a successful attempt to establish the principles of the differential and integral calculus with proper regard to modern ideas on the subject, pointing out carefully the limits under which the processes can be applied and giving examples of exceptional cases, such as Weierstrass's example of a continuous function which has no determinate derivative, and Pringsheim's example of a function for which Maclaurin's series is absolutely convergent and yet does not represent the function. One would like to see also, however, an example in which $\partial^2 z/\partial x \partial y$ and $\partial^2 z/\partial y \partial x$ are different — a case of peculiar interest in some of the applications.

The book is far more comprehensive than the usual text-books, and should therefore be useful for consultation by

students in the later years of their course. The author states that it is intended to cover about one year's work, but it seems doubtful if it could be completed in a much longer period in many colleges. However, there should be no trouble in making a judicious selection from the wealth of material and an ambitious student would be glad to possess a book which would give him so much opportunity for work outside of the classroom.

Special features of the book are the complete separation of functions of one variable from those of more than one, the liberal use of the theorems of mean value, the definition of an integral primarily as the limit of a sum, a definition to which it may be hoped all of our text-books will ultimately return, a careful introduction on variables and functions and a number of notes at the end including the examples referred to.

Where there is so much to commend it may seem invidious to criticize, but it seems to the reviewer that the definition of transfinite numbers and of infinitesimals is likely to be misunderstood by the beginner; that it would be well to print the definition of irrational numbers in larger type and let it precede the employment of such numbers; that something would be gained in the treatment of trigonometric integrals by the introduction into the text of more of the theory of the complex variable, including at least De Moivre's theorem. It is also somewhat surprising not to find a definition of curvature—the radius of curvature being defined merely as the radius of the osculating circle.

The reviewer is disposed to agree with the author in his relegation of the well-known and too much worshiped reduction formulas for the integral

$$\int x^n(a + bx^n)^v dx$$

to a note at the end of the book.

M. W. HASKELL.

American Men of Science. A Biographical Directory. Edited by J. MCKEEN CATTELL. New York, The Science Press, 1906. 4to. 7 + 364 pp.

THE chief object of this directory is to make American "men of science acquainted with one another and with one another's work." In the field which it aims to cover it is much more

comprehensive than Minerva or Who's Who in America, as it includes a tolerably complete list of sketches of those living in North America who have published any research in the natural and exact sciences, and in addition some entries of those who are held to have advanced science by teaching, by administrative work, or by the publication of text-books.

In the case of about one thousand out of the four thousand biographical notes a star is placed before the field of research. This indicates that the subject of the sketch is to be regarded as one of the thousand leading American men of science. About 80 of the 360 entries of mathematicians are distinguished in this manner. Those who are inclined to estimate the scientific standing of a man by the post which he fills will be surprised at the large number of names of younger men whose work is starred, while many who hold prominent positions are not classed among the leading scientists.

In view of the great difficulties in forming a correct estimate of the merits of the work of our contemporaries, the present undertaking appears a bold one, but the importance of accurate knowledge on this point seems to justify such ventures. All men of science and especially those who require the services of such men must feel keenly the difficulties encountered in the effort of judging the relative merits of the work of those who may be under consideration; and as such judgments are sometimes imperatively necessary, any reliable aid is a great desideratum. Moreover, there are few other things which tend to contribute so much toward substantial progress as a general feeling that the opinion of experts in regard to individual achievements will speedily become common property. The present volume appears to be a decided step towards cultivating such a feeling.

As far as possible each entry contains information on the following ten points: Name and mail address, department of investigation, place and date of birth, education and degrees, positions with dates, temporary and minor positions, honorary degrees and other scientific honors, membership in learned societies, chief subjects of research, and whether the subject of the sketch is classed with "the thousand students of the natural and exact sciences in the United States whose work is supposed to be the most important."

G. A. MILLER.

NOTES.

THE tenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of California on Saturday, September 29.

THE July number (volume 7, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the boundary value problems of linear ordinary differential equations of second order," by M. MASON; "The resolution of any collineation into perspective reflections," by M. W. HASKELL; "Linear algebras in which division is always uniquely possible," by L. E. DICKSON; "Correspondences and the theory of groups," by J. E. WRIGHT; "The trajectories of dynamics," by E. KASNER; "On the automorphic functions of the group $(0, 3; l_1, l_2, l_3)$," by R. MORRIS; "Improper multiple integrals," by R. G. D. RICHARDSON.

THE April number (volume 28, number 2) of the *American Journal of Mathematics* contains: "The motion of a solid in an infinite liquid," by A. G. GREENHILL; "The theory of implication," by BERTRAND RUSSELL.

The July number (volume 28, number 3) contains: "The geometry of differential elements of the second order with respect to the group of all point transformations," by EDWARD KASNER; "Gyroscopes and cyclones," by F. J. B. CORDEIRO; "On the primitive groups of class ten," by W. A. MANNING; "On certain unicursal twisted curves," by VIRGIL SNYDER; "Functions of three real independent variables," by H. L. COAR.

THE concluding (July) number of volume 7 of the *Annals of Mathematics* contains: "A proof of the theorem concerning artificial singularities," by D. R. CURTISS; "Another proof of the theorem concerning artificial singularities," by M. BÔCHER; "Curves of minimum moment of inertia with respect to a point," by M. MASON; "Triangles and quadrilaterals inscribed to a cubic and circumscribed to a conic," by H. S. WHITE; "On a function which occurs in the law of the mean," by E. R. HEDRICK.

THE annual list of American doctorates published in *Science* presents for the academic year 1905–1906 325 names, of which 139 are credited to the sciences. The following 9 successful candidates offered mathematics as major subject (the titles of the theses are appended): E. C. COLPITTS, Cornell, "On the twisted quintic curves"; B. F. FINKEL, Pennsylvania, "Determination of all groups of order 2 which contain cyclic self-conjugate subgroups of order 2 and whose generating operators correspond to the partitions"; C. C. GROVE, Johns Hopkins, "I. The syzygetic pencil of cubics and a new geometrical development of its Hesse group G_{216}. II. On the complete Pappus hexagon"; H. B. LEONARD, Colorado, "On the factoring of composite algebras"; J. F. MESSICK, Johns Hopkins, "Cubic curves in reciprocal triangular situation"; R. G. D. RICHARDSON, Yale, "Improper multiple integrals"; W. H. ROEVER, Harvard, "Brilliant points"; C. H. SISAM, Cornell, "Ruled surfaces of order seven having a rectilinear directrix"; G. E. WAHLIN, Yale, "The relation between the binary quadratic forms and the quadratic numerical bodies."

The number of American doctorates in mathematics for each year from 1898 to 1906 is 11, 13, 11, 18, 8, 7, 14, 21, 9.

THE first supplementary volume of the *Jahresberichte der deutschen Mathematiker-Vereinigung*, containing "Ueber die Entwicklung der Elementar-Geometrie im XIX Jahrhundert," by M. SIMON, has just been published by Teubner in Leipzig.

IN connection with the recent division of the *Proceedings of the Royal Society of London* into Series A, papers of a mathematical and physical character, and Series B, papers of a biological character, attention is called to the arrangement by which, on the advance payment of the subscription price of 15 shillings per volume, subscribers may receive the separate papers as soon as published.

SYRACUSE UNIVERSITY — The following advanced courses in mathematics are offered during the first semester of the current academic year: — By Professor W. H. METZLER: Analytic geometry, plane and solid, three hours; Newtonian potential function, three hours; Elliptic functions, three hours; Weierstrass's theory of functions, three hours. — By Professor E. D. ROE: Determinants, one hour; Theory of equations, two hours; Advanced calculus, three hours; Analytic mechanics, three

hours; Symmetric functions; three hours; Invariants, three hours; Substitutions, three hours. — By Professor W. G. BULLARD: Projective geometry, three hours; Curves and surfaces, three hours; Differential geometry, three hours.

THE various foreign universities offer courses in mathematics during the winter semester of 1906–1907 as follows:

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Elliptic functions, four hours; Seminar (with Professors Hilbert and Minkowski and Dr. Herglotz), two hours. — By Professor D. HILBERT: Mechanics of continua, four hours; Differential and integral calculus, four hours. — By Professor H. MINKOWSKI: Encyclopedia of elementary mathematics, four hours; Invariants, four hours. — By Professor C. RUNGE: Descriptive geometry, four hours; with exercises, two hours. — By Professor M. BRENDEL: Mathematics of insurance, two hours. — By Professor E. ZERMELO: Elements of analytic mechanics, four hours; Mathematical treatment of logic, one hour. — By Dr. M. ABRAHAM: Partial differential equations of mathematical physics, four hours; Seminar, two hours. — By Dr. C. CARATHÉODORY: Minimum principles of mechanics and physics, one hour. — By Dr. G. HERGLOTZ: Introduction to analytic geometry of space, four hours.

UNIVERSITY OF JENA. — By Professor J. THOMAE: Definite integrals, four hours; Differential equations, four hours; Seminar, two hours. — By Professor F. HAUSSNER: Mathematical approximations, two hours; Integral calculus with exercises, five hours; Analytic geometry of space, four hours; Seminar, one hour. — By Professor G. FREGE: Theory of functions according to Riemann, four hours. — By Professor R. RAU: Mechanics, four hours; Descriptive geometry with exercises, seven hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Analytic mechanics, four hours; Seminar, one hour. — By Professor A. MAYER: Calculus of variations, four hours. — By Professor O. HÖLDER: Differential and integral calculus, five hours; Foundations of arithmetic and numbers, two hours; Seminar, one hour. — By Professor K. ROHN: Applications of the calculus to surfaces and twisted curves, four hours; Invariants, two hours. — By Professor F. HAUSDORFF: Theory of numbers, four hours. — By Professor H. LIEBMANN: Analytic geometry of space, two hours; with exercises, one hour.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Theory of functions of a complex variable, four hours; Applications of the calculus to curves and surfaces, four hours; Groups of transformations, two hours; Seminar, one and a half hours. — By Professor A. VOSS: Algebra, four hours; Theory of algebraic curves, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Differential calculus, five hours; Theory of numbers, four hours. — By Professor K. DOEHLEMANN: Descriptive geometry, I, five hours; with exercises, three hours; Line geometry, four hours. — By Professor E. v. WEBER: Plane analytic geometry, four hours; with exercises, two hours; Integral calculus, with exercises, four hours. — By Dr. H. BRUNN: Theory of aggregates, four hours. — By Dr. F. HARTOGS: Selected chapters of the theory of functions, two hours. — By Dr. O. PERRON: — not yet announced.

OXFORD UNIVERSITY (Michaelmas Term). — By Professor W. ESSON: Analytic geometry of plane curves, two hours; Synthetic geometry of plane curves, one hour. — By Professor E. B. ELLIOTT: Sequences and series, two hours; Elementary theory of numbers, one hour. — By Professor A. E. H. LOVE: Hydrodynamics, two hours; Problems in applied mathematics, one hour. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Mr. C. E. HASELFOOT: Theory of equations, one hour. — By Mr. C. LEUDESDORF: Projective geometry, three hours. — By Mr. A. E. JOLLIFFE: Analytic geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. R. F. MCNEILE: Curve tracing, one hour. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. C. H. SAMPSON: Higher solid geometry, two hours. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours. — By Mr. E. H. HAYES: Analytical statics, three hours. — By Mr. A. L. DIXON: Hydrostatics, one hour. — By Mr. H. T. GERRANS: Tridimensional rigid dynamics, two hours. — By Mr. P. J. KIRKBY: Attractions and electrostatics, two hours.

THE royal academy of sciences at Berlin has awarded Professor F. MERTENS, of the University of Vienna, a prize of 5000 marks for his researches in the theory of cyclic equations.

PROFESSOR E. MÜLLER, of the technical school at Vienna, has been elected a corresponding member of the royal academy of sciences at Vienna.

DR. F. JUNG, docent of mathematics at the German technical school at Prague, has been transferred to a similar position at the technical school at Vienna.

PROFESSOR G. SCHEFFERS, of Darmstadt, has been appointed professor of mathematics at the technical school of Charlottenburg.

DR. L. HANNI has been appointed docent in mathematics at the University of Vienna and DR. O. PERRON at the University of Munich.

MR. B. A. HERMAN and Mr. H. W. RICHMOND have been appointed university lecturers in mathematics at Cambridge University.

AT the University of California Professor M. W. HASKELL has been promoted to a full professorship of mathematics, and Professor E. J. WILCZYNSKI to an associate professorship of mathematics.

AT Stanford University Professor H. F. BLICHFELDT has been promoted to an associate professorship of mathematics, and Dr. W. O. MENDENHALL has been appointed instructor in applied mathematics.

PROFESSOR H. P. MANNING, of Brown University, has been promoted to an associate professorship of mathematics.

AT Lehigh University Professors P. A. LAMBERT and A. E. MEAKE have been promoted to full professorships of mathematics.

AT the University of Rochester, Dr. A. S. Gale has been promoted to the newly designated Fayerweather professorship of mathematics.

PROFESSOR T. E. MCKINNEY, of Marietta College, has been appointed acting professor of mathematics at Wesleyan University.

MR. P. P. BOYD has been appointed professor of mathematics at Hanover College, Indiana.

MR. W. J. NEWLIN has been appointed associate professor of mathematics and philosophy at Amherst College.

REV. R. D. CARMICHAEL has been appointed professor of mathematics at the Presbyterian College for Men at Anniston, Alabama.

DR. J. T. ROOD has been appointed professor of mathematics at Ursinus College, Collegeville, Pennsylvania.

DR. C. C. GROVE has been appointed assistant professor of mathematics at Hamilton College, Clinton, New York.

DR. H. L. COAR, of the University of Illinois, has been appointed associate professor of mathematics at Marietta College, Marietta, Ohio.

DR. D. C. GILLESPIE has been appointed instructor in mathematics at Cornell University.

DR. G. E. WAHLIN has been appointed instructor in mathematics at the University of Iowa.

MR. J. M. THORNTON has been appointed professor of mathematics at the University of West Virginia.

DR. E. L. DODD has been appointed instructor in mathematics at the University of Illinois.

MR. E. R. SLEIGHT has been appointed professor of mathematics at Carthage College, Carthage, Illinois.

DR. W. R. LONGLEY has been appointed instructor in mathematics and astronomy at Yale University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADAM (C.). See DESCARTES (R.).

BAKER (W. M.). Algebraic geometry; a new treatise on analytical conic sections. London, Bell, 1906. 12mo. 358 pp. Cloth. 6s.

BESTHORN (R. O.). See EUCLIDES.

BOCHOW (K.). Die Funktionen rationaler Winkel; insbesondere über die Berechnung der Winkelfunktionen ohne Benutzung der trigonometrischen Reihen und der Zahl π. (Progr.) Magdeburg, 1905.

BONOLA (R.). La geometria non-euclidea: esposizione storico-critica de suo sviluppo. Bologna, Zanichelli, 1906. 8vo. 6 + 213 pp. L. 5.00

BRÖHL (J.). Doppelt unendliche Kurvenscharen, gebildet von Schraubenlinien auf den Flächen der Rotationsflächenschar $x^2 + y^2 = u^2 e^{2x}$. Münster, 1905. 8vo. 53 pp. M. 2.00

BRÜGMANN (W.). Ueber eine reell irreducible Gruppe von Berührungstransformationen. (Progr.) Hamburg, 1906. 4to. 35 pp.

BRUNN (H.). Beziehungen des du Bois-Reymondschen Mittelwertsatzes zur Ovaltheorie. Eine mathematische Studie. Berlin, Reimer, 1905. 8vo. 10 + 135 pp.

BÜCHEL (W.). Ueber die durch gewöhnliche Differentialgleichungen definierten Kurven. (Progr.) Hamburg, 1906.

BYERLY (W. E.). Harmonic functions. [2d edition.] New York, Wiley, 1906. 8vo. 57 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 5.) $1.00

CORDIER (J.). Ueber eine Gruppe von 96 Collineationen und Correlationen. (Diss.) Strassburg, 1905. 8vo. 39 pp., 2 plates.

COREY (S. A.). A method of approximation. 8vo. 4 pp. (*American Mathematical Monthly*, Vol. 8.)

DANNACHER (S.). Zur Theorie der Funktionen des elliptischen Zylinders. Frauenfeld, 1906. 4to. 38 pp. M. 2.00

DESCARTES (R.). Oeuvres, publiées sous les auspices du ministère de l'instruction publique par C. Adam et P. Tannery. (En environ 10 volumes.) Vol. VIII: Principia philosophiae. Paris, 1906. 4to. Fr. 25.00

EUCLIDES. Elementa, ex interpretatione Al-Hadschdschadschii cum commentariis Al-Narizii (Codex Leidensis 399, I), arabice et latine ediderunt notisque instruxerunt R. O. Besthorn et J. L. Heiberg. Pars I, fasciculus 2. Hafniae, 1906. 8vo. 150 pp. M. 4.80

FRÉCHET (M.). Sur quelques points du calcul fonctionnel (Thèse.) Paris, Gauthier-Villars, 1906.

FUËTER (R.). Die Theorie der Zahlstrahlen. (Hab.) Marburg, 1905. 4to. 41 pp.

GALDEANO (Z. G. DE). Tratado de análisis matematica. Vol V: Aplicaciones del cálculo infinitesimal al estudio de las figuras en el espacio. Zaragoza, Casañal, 1905. 8vo. 576 pp. (Nueva enciclopedia matemática, Vol. 8.) P. 8.00

GASSER (A.). Ueber die Nullstellen der Besselschen Funktionen. (Diss.) Bern, Wyss, 1904. 8vo. 48 pp., 2 plates.

GROSSMANN (M.). Analytische Geometrie. Leitfaden für den Unterricht an höheren Lehranstalten. Basel, Helbing, 1906. 8vo. 104 pp. Boards. M. 2.00

HABICHT (C.). Die Steinerschen Kreisreihen. (Diss.) Bern, Wyss, 1904. 8vo. 37 pp., 4 plates.

HALSTED (G. B.). Synthetic projective geometry. [2d edition.] New York, Wiley, 1906. 8vo. 37 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward No. 2.) $1.00

HEIBERG (J. L.). See EUCLIDES.

HIMSTEDT (A.). Ueber Cartesische Ovale. (Progr.) Nordhausen, 1906. 8vo. 27 pp.

HINTON (C. H.). The fourth dimension. London, Sonnenschein, 1906. 12mo. 278 pp. Cloth. 4s. 6d.

HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. Heft 2: Die Kegelschnitte. Abteilung 1, A: Aufgaben. 3te, vermehrte Auflage, bearbeitet von O. Jahn und F. Hochheim. Leipzig, Teubner, 1906. 8vo. 90 pp. M. 1.80

HOCHHEIM (F.). See HOCHHEIM (A.).

HOFF (H.). Die Eigenschaften der Ordnungskurven einer beliebigen Reciprocität in der Ebene. (Diss.) Halle, Kaemmerer, 1906. 8vo. 63 pp., 2 plates.

HUNTINGTON (E. V.). The continuum as a type of order; an exposition of the modern theory, with an appendix on the transitive numbers. Cambridge, Mass., Harvard University, 1906.

HYDE (E. W.). Grassmann's space analysis. [2d edition.] New York, Wiley, 1906. 8vo. 51 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 6.) $1.00

ILNICKI (E.). Ueber die Prinzipien der Infinitesimalrechnung und über die Wandlungen, welche die Darstellung dieses Zweiges der Mathematik im Laufe seiner Entwicklung erfahren hat. (Progr.) Czernowitz, 1905. 8vo. 21 pp., 1 plate.

JAHN (O.). See HOCHHEIM (A.).

JOHNSON (W. W.). Differential equations. [2d edition.] New York, Wiley, 1906 8vo. 71 pp. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 9.) $1.00

JOUFFRET (E.). Mélanges de géométrie à quatre dimensions. Paris, Gauthier-Villars, 1906. 8vo. 11 + 228 pp. Fr. 7.50

JUNKER (F.). Repetitorium und Aufgabensammlung zur Integralrechnung. 2te, verbesserte Auflage. Leipzig, Göschen, 1906. 16mo. 135 pp. Cloth. (Sammlung Göschen, No. 147.) M. 0.80

KIEFER (C. L.). Ueber Strahlenkongruenzen zweiter Klasse fünfter und niedrigerer Ordnung. (Diss.) Strassburg, 1905. 8vo. 40 pp.

KLEIN (F.). Autographierte Vorlesungshefte. Ueber lineare Differentialgleichungen der zweiten Ordnung, ausgearbeitet von E. Ritter. (Neuer, unveränderter Abdruck.) Leipzig, Teubner, 1906. 4 + 524 pp. M. 8.50

LARMOR (J.). See POINCARÉ (H.).

LESSER (O.). Die Infinitesimalrechnung im Unterrichte der Prima. Berlin, Salle, 1906. 8vo. 121 pp. M. 1.60

LIPPMANN (A.). Die absolute Wahrheit der Euklidischen Geometrie. Eine kritische Untersuchung der Grundlagen der Geometrie. Beweise für die Wahrheit der Axiome und Postulate, insbesondere für die des Parallelenaxioms (V. Postulat Euklids). Leipzig, Gerstäcker, 1906. 8vo. 68 pp. M. 3.60

LUCAS DE PESLOUAN (C.). N.-H. Abel. Sa vie et son œuvre. Paris, Gauthier-Villars, 1906. 12mo. 13 + 170 pp., portrait. Fr. 5.00

MACFARLANE (A.). Vector analysis and quaternions. [2d edition.] New York, Wiley, 1906. 8vo. 33 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 8.) $1.00

MCMAHON (J.). Hyperbolic functions. [2d edition.] New York, Wiley, 1906. 8vo. 62 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 4.) $1.00

MANDLE (J.). Kurzgefasstes Lehrbuch der Mathematik für Ingenieure. Wien, Lehman & Wentzel, 1906. 8vo. 8 + 327 pp. M. 9.50

MATHÉ-REICHENBERG (J.). Karl Friedrich Gauss. Leipzig, 1906. 8vo. Portrait. M. 1.00

MAUDERLI (S). Die Interpolation und ihre Verwendung bei der Benutzung und Herstellung mathematischer Tafeln. Solothurn, 1906. 8vo. 147 pp. M. 3.60

MERRIMAN (M.). The solution of equations. [2d edition.] New York, Wiley, 1906. 8vo. 32 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 10.) $1.00

MEYER (P.). Beweis eines von Euler entdeckten Satzes, betreffend die Bestimmung von Primzahlen. (Diss.) Strassburg, 1906. 8vo. 26 pp.

MORALE (M.). Sopra un modo di generare il complesso quadratico di rette. Caltanissetta, Castaldi-Petrantoni, 1906. 8vo. 15 pp.

MÜHLEMANN (F.). Enveloppen der Euler'schen Geraden. (Diss.) Bern, Wyss, 1905. 8vo. 75 pp., 8 plates. M. 3.00

PASQUIER (L. G. DU). Zahlentheorie der Tettarionen. (Diss.) Zürich, 1906. 8vo. 76 pp.

PIEL (C.). Ueber die Kegelschnitte, welche durch drei Punkte und zwei Tangenten oder durch zwei Punkte und drei Tangenten bestimmt sind, und die Kegelschittsysteme. (Diss.). Strassburg, 1905. 8vo. 73 pp., 6 plates.

PINCHERLE (S.). Lezioni di algebra complementare, dettate nella r. Università di Bologna e redatte per uso degli studenti. Analisi algebrica. Bologna, Zanichelli, 1906. 8vo. 366 pp. L. 10.00

PITTARD-BULLOCK (H. A. P.). The power of the continuum. (Diss., Rostock.) Berlin, Ebering, 1905. 8vo. 42 pp.

POINCARÉ (H.). Der Wert der Wissenschaft. Autorisierte deutsche Ausgabe von E. Weber. Mit Anmerkungen von H. Weber. Leipzig, Teubner, 1906. 8vo. 240 pp.

POINCARÉ (H.). Science and hypothesis. With a preface by J. Larmor. London, 1905. 12mo. 244 pp. Cloth. $1.50

RITTER (E.). See KLEIN (F.).

ROSE (E.). Die Axiome der projektiven Geometrie linearer Mannigfaltigkeiten. (Diss.) Strassburg, 1905. 8vo. 58 pp.

SCHAUFF (P.). Ueber die geodätischen Linien auf einem Kegel. (Diss.) Münster, 1906. 8vo. 44 pp. M. 1.50

SCHLEIERMACHER (L.). Potenz und Kegelschnitt. (Progr.) Aschaffenburg, 1906.

SCHULTZ (E.). Die Transformation der Ausdrücke $[\phi, \psi]$ in solche, deren Variablen Bedingungsgleichungen erfüllen. (Progr.) Stettin, 1906. 4to. 16 pp.

SCHWARZLOSE (R.). Ueber eine geometrische Beziehung der Fresnelschen Wellenfläche zum dreiaxigen Ellipsoid. (Diss.) Rostock, 1906. 8vo. 50 pp. M. 1.50

SENFTNER (G.). Ein mechanisches Problem aus der Variationsrechnung. (Diss.) Rostock, 1905. 8vo. 32 pp.

SERVAIS (C.). Cours de géométrie analytique de la Faculté des sciences. Gand, 1906. 4to. Vol. I: 188 pp.; Vol. II: 188 pp. (Autographié.) Fr. 12.50

SHEARMAN (A. T.). Development of symbolic logic; a critical-historical study of the logical calculus. London, Williams & Norgate, 1906. 12mo. 254 pp. Cloth. 5s.

SIMON (M.). Analytische Geometrie des Raumes. 2te, verbesserte Auflage. 2ter Abdruck. Leipzig, Göschen, 1906. 16mo. 205 pp. Cloth. (Sammlung Göschen, No. 89.) M. 0.80

SMITH (D. E.). History of modern mathematics. [2d edition.] New York, Wiley, 1906. 8vo. 63 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 1.) $1.00

SOLER (E.). Sulle proiezioni merisogone. Messina, d'Amico, 1906. 8vo. 43 pp.

STECKELBERG (H.). Die Elemente der Differential- und Integalrechnung. Leipzig, Teubner, 1906. 8vo. 48 pp. M. 0.80

STEINGRÄBER (W.). Ueber partielle Differentialgleichungen erster Ordnung im R_4. (Diss.) Greifswald, 1906. 8vo. 46 pp.

SYRUP (F.). Astigmatische Spiegelung im dreiaxigen Ellipsoid. (Diss.) Rostock, 1905. 8vo. 45 pp., 2 plates.

TANNERY (P.). See DESCARTES (R.).

THORNBURG (C. L.). Calculus notes. Lancaster, Pa., New Era Printing Co., 1906. 12mo. 33 pp. Cloth. $0.50

VOCKERODT (L.). Ueber die Entwickelung von Thetaquotienten in unendliche Reihen. (Diss.) Strassburg, 1905. 8vo. 27 pp.

WEBER (E.) and (H.). See Poincaré (H.).

WEINBAUM (O.). Die Spiegelung einer unendlichen Ebene in einem zu ihr senkrechten elliptischen Zylinder. (Diss.) Berlin, Mayer & Müller, 1906. 8vo. 50 pp., 3 plates. M. 1.80

WELD (L. G.). Determinants. [2d edition.] New York, Wiley, 1906. 8vo. 37 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 3.) $1.00

WHITEHEAD (A. N.). Mathematical concepts of the material world. London, Dulau, 1906. 4ts. 60 pp. 2s. 6d.

WILCZYNSKI (E. J.). Projective differential geometry of curves and ruled surfaces. Leipzig, Teubner (New York, Stechert), 1906. 8vo. 8 + 298 pp. Cloth. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. XVIII.) M. 10.00

WOODWARD (R. S.). Probability and theory of errors. [2d edition.] New York, Wiley, 1906. 8vo. 41 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 7.) $1.00

ZAHRADNIECK (K.). Zur Frage der Einführung der Infinitesimalrechnung an den österreichischen Mittelschulen. Wien, 1905. 8vo. 48 pp. M. 1.80

ZEHME. Eigenschaften des Krümmungsschwerpunktes ebener Kurven. (Progr.) Arnstadt, 1906. 4to. 18 pp.

ZINDLER (K.). Liniengeometrie mit Anwendungen. Vol. II. Leipzig, Göschen, 1906. 12mo. 252 pp. Cloth. (Sammlung Schubert, No. 50.) M. 8.00

II. ELEMENTARY MATHEMATICS.

BACCINI (G.). Polisezione dell' angolo rettilineo, ossia regola facile ed infallibile per dividere qualunque angolo rettilineo in qualsivoglia numero di parti uguali. Reggio-Emilia, Artigianelli, 1906. 8vo. 14 pp.

BIEL (B.). Mathematische Aufgaben für die höheren Lehranstalten, unter möglichster Berücksichtigung der Anwendungen, wie überhaupt der Verknüpfung der Mathematik mit anderen Gebieten zusammengestellt. 2ter Teil: Die Oberstufe. Leipzig, Freytag, 1906. 8vo. 299 pp. Cloth. M. 3.50

BLOCK (C.). Lehr- und Uebungsbuch für den planimetrischen Unterricht an höheren Schulen. Teil 2: Untertertia. Leipzig, Teubner, 1906. 8vo. M. 0.80

BÖKELER (A.). Stereometrische Aufgaben aus den Reifeprüfungen der Gymnasial-Abiturienten. Auflösungen. Ravensburg, Alber, 1906. 8vo, 4 + 80 pp. M. 2.00

BORK (H.), CRANTZ (P.) und HAENTZSCHEL (E.). Mathematischer Leitfaden für Realschulen. 5te, verbesserte Auflage. Leipzig, Dürr, 1906.

BORTOLOTTI (E.). Nozioni di geometria per le scuole tecniche: guida ad esami di geometria. Roma, Voghera, 1906. 8vo. 280 pp.

BREMIKER (C.). Logarithmisch-trigonometrische Tafeln mit 5 Decimalstellen. 10te Auflage, von A. Kallius. Berlin, 1906. 8vo. 192 pp. Cloth. M. 1.50

——. See VEGA (G. VON).

BROOKSMITH (E. J.). Mathematical papers for admission to the Royal Military Academy and Royal Military College, 1896-1905. London, Macmillan. 12mo. Cloth. 6s.

CAREY (F. S.). Elementary solid geometry. Enlarged edition. London, Arnld, 1906. 12mo. 124 pp. Cloth. 2s. 6d.

CRANTZ (P.). See BORK (H.).

DELILLE (A.). See LEGENDRE.

DUPUIS (J.). Tables de logarithmes à cinq décimales contenant les logarithmes des nombres entiers de 1 à 10,000, les logarithmes des sinus et des tangentes calculés de minute en minute jusqu'à 90 degrés, plusieurs tables usuelles et un grand nombre de formules et de nombres utiles. 28e édition. Paris, Hachette, 1906. 16mo. 4 + 231 pp. Fr. 2.50

GAJDEČZKA (J.). Uebungsbuch zur Arithmetik und Algebra für die oberen Klassen der Mittelschulen. 7te Auflage. Wien, Tempsky, 1906. 8vo. 230 pp. Cloth. M. 3.00

GLASER (R.). Stereometrie. 2te, umgearbeitete und vermehrte Auflage. Neudruck. Leipzig, Göschen, 1906. 16mo. 205 pp. Cloth. (Sammlung Göschen, No. 89.) M. 0.80

GRÉVY (A.). Géométrie plane (conforme au programme du 27 juillet 1905) à l'usage des élèves du premier cycle A (classes de quatrième A et troisième A). 3e édition. Paris, Vuibert et Nony, 1906. 16mo. 8 + 283 pp. Fr. 2.00

GRÜTTNER (A.). Dreieckskonstruktionen, bei denen 3 Punkte gegeben sind. (Progr.) Wollstein, 1906. 8vo. 30 pp.

GUGLIELMI (A.). Nozioni di geometria per le scuole tecniche. Quarta edizione modificata e corredata di molti esercizi. Torino, Paravia, 1906. 8vo. 206 pp. L. 2.50

GUILLEMIN (A.). Tableaux logarithmiques A et B équivalant à des tables de logarithmes à 6 et à 9 décimales et notice explicative donnant la théorie et le mode d'emploi des tableaux. Paris, Alcan, 1906. 8vo. 32 pp.

HAENTZSCHEL (E.). See BORK (H.).

HALL (H. S.) and STEVENS (F. H.). School geometry, parts 1 and 2. Introductory course of experimental and practical work. London, Macmillan, 1906. 12mo. 150 pp. Cloth. 2s. 6d.

HALLER VON HALLERSTEIN (F.). Lehrbuch der Elementar-Mathematik. Für die Portepeefähnrichs-Prüfung in der k. preussischen Armee und die Prüfung zum Eintritt in die k. Marine bearbeitet. 11te Auflage. Herausgegeben und für den Gebrauch in der Prima der Gymnasien und Realgymnasien erweitert von B. Hülsen. Teil 2: Geometrie. Berlin, Nauck, 1906. 8vo. 8 + 552 pp. M. 6.80

HIRSCH (H.). Ueber gerade und ungerade Permutationen. (Progr.) Oppenheim, 1906. 4to. 10 pp.

HOČEVAR (F.). Lehrbuch der Geometrie nebst einer Sammlung von Uebungsaufgaben für Oberrealschulen. 2te Auflage. Wien, Temspky, 1906. 8vo. 330 pp. Cloth. M. 4.00

HÖFLER (A.). Vorschläge zu einer zeitgemässen Umgestaltung des mathematischen Unterrichts an den österreichischen Gymnasien und Realschulen. Leipzig, Teubner, 1906. 8vo. 3 + 15 pp.

HÜLSEN (B.). See HALLER VON HALLERSTEIN (F.).

INSKIP (G. D.). Tables of five place squares and logarithms of feet, inches and 32ds of inches from zero to 100 feet; logarithmic secants for hip, valley, and rafter from zero to 18-inch rise to 1-foot base, advancing by 32ds; logarithmic functions, natural sines and co-sines for every minute of the quadrant. New York, Myron C. Clark Publishing Co., 1906. 12mo. 2 + 277 pp. Limp leather. $3.00

KALLIUS (A.). See BREMIKER (C.).

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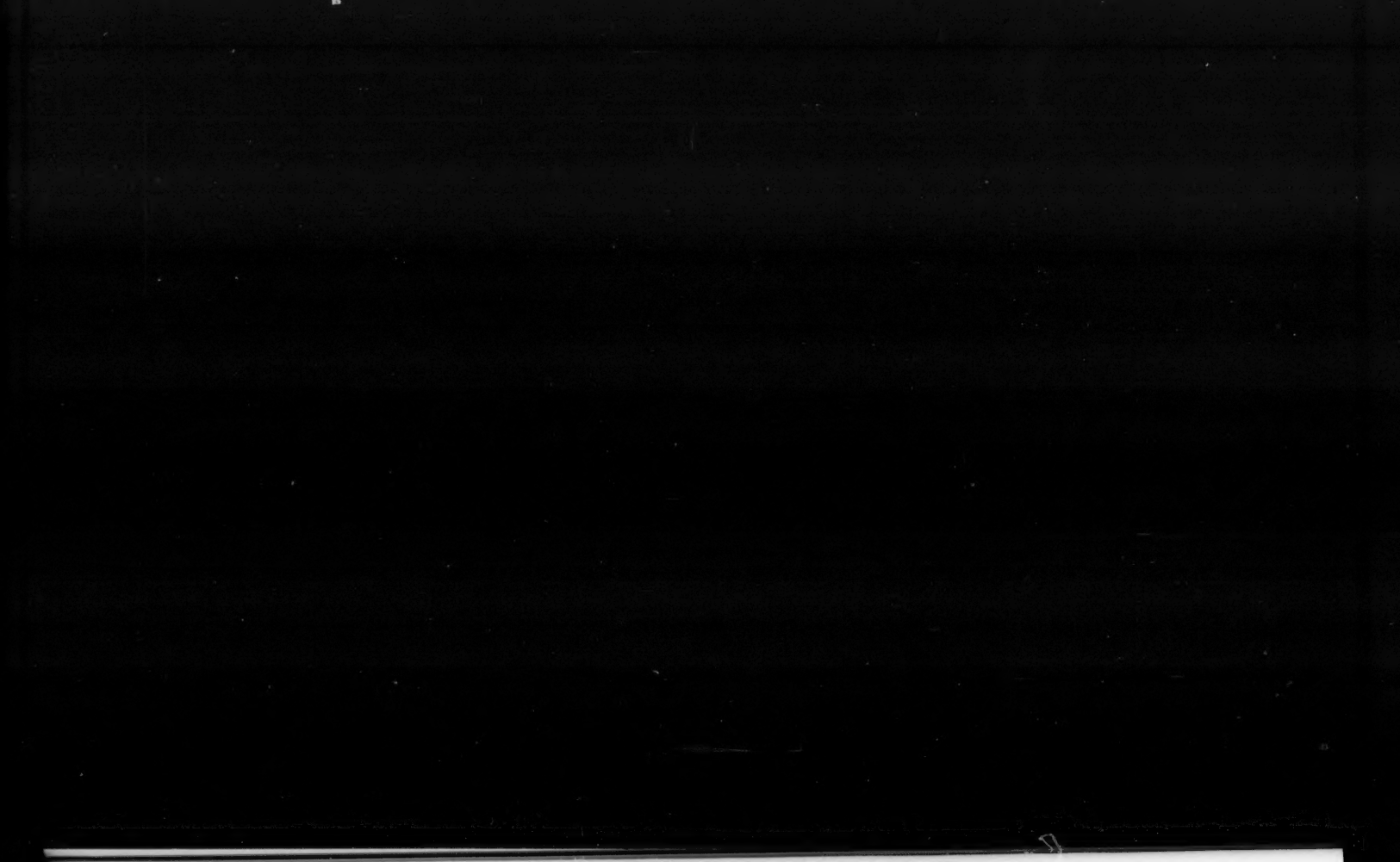
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